Mechanics 2
Statics of Bodies
Statics of Rigid Bodies

RECAP OF M1..............................................................................................................................................2
EQUILIBRIUM OF RIGID BODIES........................................................................................................4
EXAM QUESTION........................................................................................................................................7
LIMITING EQUILIBRIUM.........................................................................................................................9
PROBLEMS INVOLVING LADDERS......................................................................................................12
CLIMBING LADDERS............................................................................................................................17
QUESTIONS.............................................................................................................................................19
Recap of M1

It is always best to start with an example from the previous unit to remind you of the basics.

Example 1
A body of mass 5 kg is held in equilibrium under gravity by two inextensible light ropes. One rope is horizontal and the other is inclined at an angle $\theta$ to the horizontal, as shown in the diagram below. The tension in the rope inclined at $\theta$ to the horizontal is 72N.

Find
a) the angle $\theta$, giving your answer to the nearest degree.
b) the tension $T$ in the horizontal rope, giving your answer to the nearest N.

a) Most of the questions from the moments section in M1 can be solved by simply resolving forces. The body is in equilibrium hence the vertical component of the 72N force must equal $5g$.

\[ 72 \sin \theta = 5g \quad (1) \]
\[ \theta = 42.88^\circ \]
\[ \theta = 43^\circ \]
b) The horizontal component of the 72N force must equal T.

\[ 72 \cos \theta = T \]

\[ 72 \times \cos 43^\circ = T \]

\[ T = 53N \]

The moment of a force about a particular point is:

(force \times \text{perpendicular distance})

\[ \text{Moment about Q} = F \times d \]

Remember that the unit is Nm.

Algebraic sum of moments

If a number of coplanar forces act on a rigid body then their moments about a given point may be added. Take great care with the direction of rotation
Equilibrium of Rigid Bodies

A rigid body is said to be in equilibrium if;

1. the vector sum of the forces acting is zero (the sum of the components in any direction is zero)
2. the algebraic sum of the moments of the force about a particular point is zero.

All problems within this unit need to be attacked in the same fashion.

1. Draw a clearly labeled diagram showing all forces.
2. Resolve the forces perpendicularly and vertically remembering to set them equal to zero. On some occasions the forces will need to be resolved parallel and perpendicular to a plane.
3. Take moments about a convenient point and equate the algebraic sum of the moments to zero. The aim is to choose a point that has only a few unknowns. If you choose a point that makes things more difficult simply select another point.

Example 2
A uniform rod \( AB \) of mass 12 kg and length 15 m is smoothly hinged at \( A \) and has a particle of mass 28 kg attached to it at \( B \). A light inextensible string is attached to the rod at the point \( C \) where \( AC = 9 \text{ m} \) and to the point \( D \) vertically above \( A \), keeping the rod in a horizontal position. The tension in the string is \( TN \). If the angle between the rod and the string is \( 25^\circ \), calculate, in terms of \( T \), the resultant moment about \( A \) of the forces acting on the rod.
So by taking moments about A, one needs to consider, the weight of the rod, the particle placed at B and the tension in the string acting at C.

\[(12g \times 7.5) + (28g \times 15) - (9 \times T \sin 25^\circ)\]

\[(510g - 3.80T)\text{Nm}\]

**Example 3**
A non-uniform rod PQ of mass 12kg and length 8m rests horizontally in equilibrium, supported by two strings attached at the ends P and Q of the rod. The strings make angles of 45° and 60° with the horizontal as shown in the diagram.

(a) Obtain the tensions in each of the strings.
(b) Determine the position of the centre of mass of the rod.

![Diagram of the non-uniform rod](image)

a) Resolving horizontally gives:

\[T_1 \cos 45^\circ = T_2 \cos 60^\circ\]

\[T_1 \frac{\sqrt{2}}{2} = T_2 \frac{1}{2}\]

\[\sqrt{2}T_1 = T_2\]

Resolving vertically gives:
\[ T_1 \sin 45^\circ + T_2 \sin 60^\circ = 12g \]
\[ T_1 \frac{\sqrt{2}}{2} + T_2 \frac{\sqrt{3}}{2} = 12g \]
\[ \sqrt{2}T_1 + \sqrt{3}T_2 = 24g \]

By substituting back in for \( T_2 \) this gives:
\[ \sqrt{2}T_1 + \sqrt{2}\sqrt{3}T_1 = 24g \]
\[ T_1(\sqrt{2} + \sqrt{6}) = 24g \]
\[ T_1 = \frac{24g}{(\sqrt{2} + \sqrt{6})} = 60.87N \]

Therefore
\[ T_2 = 86.09N \]

b) Taking moments about Q gives:
\[ 12gx = 8T_1 \cos 45^\circ \]
\[ 12gx = 4T_1 \sqrt{2} \]
\[ x = 2.93m \]

One of the fundamental ideas to remember with this type of problem is that surds must be used until the final calculation. This example may, at first, appear tricky but there will be others to practice on later.
Exam Question

A uniform rod $AB$, of length $8a$ and weight $W$, is free to rotate in a vertical plane about a smooth pivot at $A$. One end of a light inextensible string is attached to $B$. The other end is attached to point $C$ which is vertically above $A$, with $AC = 6a$. The rod is in equilibrium with $AB$ horizontal, as shown below.

(a) By taking moments about $A$, or otherwise, show that the tension in the string is $5/6 W$.

Add the forces to the diagram.

Taking moments about $A$ gives:

$$4aW = 8aTSinB$$

By Pythagoras $CB = 10a$

$$\therefore SinB = \frac{6a}{10a} = \frac{3}{5}$$
Hence

\[ 4aW = 8aT \times \frac{3}{5} \]

\[ W = 2T \times \frac{3}{5} \]

\[ \therefore T = \frac{5}{6}W \]

\((b)\) Calculate the magnitude of the horizontal component of the force exerted by the pivot on the rod.

Resolving forces horizontally.

\[ X = TC\cos B \]

\[ \cos B = \frac{8a}{10a} = \frac{4}{5} \]

\[ T = \frac{5}{6}W \]

Therefore

\[ X = \frac{2}{3}W \]
Limiting Equilibrium

If a body is in limiting equilibrium then one of the forces acting must be friction. The condition to remember is that \( F \leq \mu R \). At the instant that motion is about to take place friction will have its highest value of \( \mu R \).

Example 4
A smooth horizontal rail is fixed at a height of 3m above a horizontal playground whose surface is rough. A straight uniform pole AB, of mass 20kg and length 6m, is placed to rest at point C on the rail with the end A on the playground. The vertical plane containing the pole is at right angles to the rail. The distance AC is 5m and the pole rests in limiting equilibrium. Calculate:

a) the magnitude of the force exerted by the rail on the pole, giving your answer to 3 sig fig.

b) the coefficient of friction between the pole and the playground, giving your answer to 2 decimal places.

c) the magnitude of the force exerted by the playground on the pole, giving your answer to the nearest N.
a) Adding forces to the diagram:

Taking moments about A gives:

\[ 5S = 20g \times 3 \times \cos A \]

\[ \cos A = 0.8 \]

Therefore

\[ S = 94N \]

b) Limiting equilibrium therefore \( F = \mu R \).

Resolving horizontally gives:

\[ S \times \sin A = F \]

\[ S \times 0.6 = F \]

\[ F = 56.4N \]

Resolving vertically gives:

\[ S \times \cos A + R = 20g \]

\[ 94 \times 0.8 + R = 196 \]
R = 120.8N

F = \mu R

Therefore \mu = 0.47

c) Magnitude of the force exerted by the ground on the pole is given by:

$$= \sqrt{(120.8^2 + 56.4^2)}$$

$$= 133N$$
Problems Involving ladders

Ladders will either be lent against a wall or horizontal. We sometimes have to consider frictional forces on the ladder due to the floor or wall (ladder is in contact with a 'rough' surface). Remember that the friction $F$ acts parallel to the surface in such a direction as to oppose the motion.

Example 5
A uniform ladder of mass 30kg rests against a smooth vertical wall with its lower end on rough ground (coefficient of friction 0.25), and its top against a smooth vertical wall. The ladder rests at an angle of $60^\circ$ to the horizontal. Find the magnitude of the minimum horizontal force required at the base to prevent slipping.

We need to find $S$, $V$, $R$ and $F$.

The easiest one to find first is $S$.

Taking moments about Q gives:

$$S \times L \sin 60^\circ = 30g \times 0.5L \cos 60^\circ$$

$$S = \frac{30g \times 0.5 \cos 60^\circ}{\sin 60^\circ}$$

$$S = 84.87 \text{N}$$
Resolving vertically gives:

\[ R = 30g \]

Resolving horizontally gives:

\[ S = F + V \]

Given that \( F = \mu R \)

\[ 84.87 = 0.25 \times 30 \times g + V \]

\[ 84.87 = 73.5 + V \]

\[ V = 11.4N \]

What is the maximum horizontal force that could be applied at the base of the ladder without slipping occurring?
In this situation friction is acting in the opposite direction.

Therefore

\[ S + F = V \]

\[ 84.87 + 73.5 = V \]

\[ V = 158.4N \]
Example 6
The diagram shows a ladder AB of mass 8kg and length 6m resting in equilibrium at an angle of 50° to the horizontal with its upper end A against a smooth vertical wall and its lower end B on rough horizontal ground, coefficient of friction μ. Find forces S, F and R and the least possible value of μ if the centre of gravity of the ladder is 2m from B.

Taking moments about B gives:

\[ S \times 6 \sin 50° = 8g \times 2 \cos 50° \]

\[ S = 21.9 \text{N} \]

Resolving horizontally gives:

\[ S = F \]

\[ F = 21.9 \text{N} \]

Resolving vertically gives:

\[ R = 8g = 78.4 \text{N} \]
Least value of $\mu$ occurs when $F = \mu R$

$$\mu = \frac{F}{R}$$

$$\mu = \frac{21.9}{78.4} = 0.28$$

The following example considers friction on the floor and the wall.

**Example 7**
A uniform ladder of mass 25kg and length $L$ rests against a rough vertical wall (coefficient of friction $\mu = \frac{1}{3}$) with its base on rough ground (coefficient $\alpha = \frac{1}{5}$) and it makes an angle of $61^\circ$ with the ground. Find the magnitude of the minimum horizontal force that must be applied to the base in order to prevent slipping.

Taking moments about $Q$ gives:

$$S \times L \sin 61^\circ + W \times L \cos 61^\circ = 12.5g \times L \times \cos 61^\circ$$

$$S \times \sin 61^\circ + W \times \cos 61^\circ = 12.5g \times \cos 61^\circ \ (1)$$
Using $W = \mu S$ equation (1) becomes:

\[ S \times \sin 61^\circ + \frac{S}{3} \times \cos 61^\circ = 12.5g \times \cos 61^\circ \]

\[ S \left(\sin 61^\circ + \frac{\cos 61^\circ}{3}\right) = 12.5g \times \cos 61^\circ \]

\[ S = \frac{12.5g \times \cos 61^\circ}{\sin 61^\circ + \frac{\cos 61^\circ}{3}} = 57.31N \]

Resolving vertically gives:

\[ W + R = 25g \]
\[ \mu S + R = 25g \]

\[ R = 25g - \left(\frac{57.31}{3}\right) \]

\[ R = 225.9N \]

Using $F = \alpha R$

\[ F = 0.2 \times 225.9 \]

\[ F = 45.18N \]

Finally, resolving horizontally gives:

\[ F + V = S \]
\[ 45.18 + V = 57.31 \]

\[ V = 12.1N \]

Therefore the minimum horizontal force to prevent slipping is 12.1N
Climbing ladders

Obviously safety is the ultimate concern when climbing a ladder. In deciding whether it is safe to climb to the top of a ladder one has to consider the magnitude of the frictional force acting on the ladder. This in itself is dependent on the roughness of the ground. If a MAN is already on a ladder and the system is in limiting equilibrium then any further movement up the ladder will cause it to slip. The example below considers such a situation.

Example 8
A uniform ladder of mass 30kg and length 10m rests against a smooth vertical wall with its lower end on rough ground. The coefficient of friction between the ground and the ladder is 0.3. The ladder is inclined at an angle \( \theta \) to the horizontal where \( \tan \theta = 2 \). Find how far a boy of mass 30 kg can ascend the ladder without it slipping.

Assume that the boy can climb a height \( y \) m up the ladder.

![Diagram of ladder and boy](image)

Taking moments about B gives:

\[
S \times 10 \times \sin \theta = 30g \times y \times \cos \theta + 30g \times 5 \times \cos \theta
\]

\[
10S \times \sin \theta = 30g(y + 5)\cos \theta
\]

\[
10S \times \tan \theta = 30g(y + 5)
\]

\[
20S = 30g(y + 5)
\]
Resolving vertically gives:

\[ R = 60g \]

Given that \( F = \mu R \), \( F = 0.3 \times 60g = 18g \)

Resolving horizontally gives:

\[ F = S \]

Therefore \( S = 18g \)

Using \[ 20S = 30g(y + 5) \]

\[ 20 \times 18g = 30g(y + 5) \]

\[ 12 = y + 5 \]

\[ y = 7 \]

Therefore the boy can climb 7m up the ladder.
Questions

1. A uniform rod of mass $M$ rests in limiting equilibrium with the end $A$ standing on rough horizontal ground and the end $B$ resting against a smooth vertical wall. The vertical plane containing $AB$ is perpendicular to the wall. The coefficient of friction between the rod and the ground is 0.2. Find, to the nearest degree, the angle at which the rod is inclined to the vertical.

2. A uniform rod of mass $M$ rests in limiting equilibrium with the end $A$ standing on rough horizontal ground and the end $B$ resting against a smooth vertical wall. The vertical plane containing $AB$ is perpendicular to the wall. The coefficient of friction between the rod and the ground is 0.75. Given that the ladder makes an angle $\alpha$ with the floor show that

$$\tan \alpha = \frac{2}{3}$$

3. A non uniform ladder $AB$ of length 15m and mass 40kg has its centre of gravity at a point 5m from $A$. The ladder rests with end $A$ on rough horizontal ground (coefficient of friction 0.25) and end $B$ against a rough vertical wall (coefficient of friction 0.2). The ladder makes an angle $\alpha$ with the horizontal such that

$$\tan \alpha = \frac{9}{4}$$

A straight horizontal string connects $A$ to a point at the base of the wall directly below $B$. A man of mass 80kg begins to climb the ladder. How far up the ladder can the man climb without causing tension in the string? What tension must the string be capable of withstanding if the man is to climb to the top of the ladder?
4. The diagram below shows a uniform ladder AB of length 2a and mass \( m \), with the end A resting on a rough horizontal floor.

The ladder is held at an angle \( \theta \) to the vertical by means of a light inextensible rope attached to the point N, where AN = 1.5a. The other end of the rope is attached to a point C, which is at a height 3a vertically above the end A of the ladder. By taking moments about C find the magnitude of the force of friction acting on the ladder at A. Also calculate the magnitude of the vertical component of the reaction at A.

Given that the coefficient between the floor and the ladder is \( \frac{1}{\sqrt{3}} \), show that when the ladder is on the point of slipping at A its inclination to the vertical is given by \( \theta = \frac{\pi}{3} \).
The figure shows a uniform rod $AB$ of weight $W$ resting with one end $A$ against a rough vertical wall. One end of a light inextensible string is attached at $B$ and the other end is attached at a point $C$, vertically above $A$. The points $A$, $B$ and $C$ lie in the same vertical plane with $AB = BC = 4a$ and $AC = a$. If the system is in limiting equilibrium, calculate:

a) the tension in the string
b) the angle that the rod makes with the horizontal.
c) the magnitude of the resultant force acting at $A$. 

![Diagram of the system](image-url)
6 The diagram below shows a uniform rod AB, of weight W being held in limiting equilibrium at an angle of 30° to a rough plane by a vertical string at the point B. The end A is in contact with a rough surface inclined at an angle of 30° to the horizontal. Find T in terms of W and calculate the coefficient of friction between the rod and the inclined plane.

7 The diagram below shows a uniform rod at rest in limiting equilibrium on a rough peg at A and a smooth peg at C. Given that MC = CB, find the coefficient of friction at A and the normal reaction at C (in terms of W).
The diagram below shows a heavy uniform rod of length $2a$ and mass $m$, resting in equilibrium with its two ends on two smooth surfaces. The normal reactions at the ends of the rods have magnitudes $R$ and $S$. The inclinations of the planes to the horizontal are $\frac{\pi}{6}$ and $\frac{\pi}{4}$ and the rod lies in the vertical plane containing lines of greatest slope of both planes.

a) Show that $R = S\sqrt{2}$.

b) By taking moments about the centre of the rod, prove that the inclination of the rod to the horizontal is $\cot^{-1}(1 + \sqrt{3})$. The identities for $\cos(A + B)$ and $\cos(A - B)$ may be useful.
1. A uniform ladder $AB$, of mass $m$ and length $2a$, has one end $A$ on rough horizontal ground. The coefficient of friction between the ladder and the ground is 0.5. The other end $B$ of the ladder rests against a smooth vertical wall. The ladder rests in equilibrium in a vertical plane perpendicular to the wall, and makes an angle of $30^\circ$ with the wall. A man of mass $5m$ stands on the ladder which remains in equilibrium. The ladder is modelled as a uniform rod and the man as a particle. The greatest possible distance of the man from $A$ is $ka$.

Find the value of $k$.

June 2001, Q3

2. Figure 2

Figure 2 shows a horizontal uniform pole $AB$, of weight $W$ and length $2a$. The end $A$ of the pole rests against a rough vertical wall. One end of a light inextensible string $BD$ is attached to the pole at $B$ and the other end is attached to the wall at $D$. A particle of weight $2W$ is attached to the pole at $C$, where $BC = x$. The pole is in equilibrium in a vertical plane perpendicular to the wall. The string $BD$ is inclined at an angle $\theta$ to the horizontal, where $\sin \theta = \frac{3}{5}$. The pole is modelled as a uniform rod.

(a) Show that the tension in $BD$ is $\frac{5(5a - 2x)}{6a}W$.

(b) $x$ in terms of $a$.

The vertical component of the force exerted by the wall on the pole is $\frac{7}{6}W$. Find
A straight log $AB$ has weight $W$ and length $2a$. A cable is attached to one end $B$ of the log. The cable lifts the end $B$ off the ground. The end $A$ remains in contact with the ground, which is rough and horizontal. The log is in limiting equilibrium. The log makes an angle $\alpha$ to the horizontal, where $\tan \alpha = \frac{5}{12}$. The cable makes an angle $\beta$ to the horizontal, as shown in Fig. 3. The coefficient of friction between the log and the ground is 0.6. The log is modelled as a uniform rod and the cable as light.

(a) Show that the normal reaction on the log at $A$ is $\frac{2}{5}W$.

(b) Find the value of $\beta$.

The tension in the cable is $kW$.

(c) Find the value of $k$. 

Jan 2002, Q5

June 2002, Q7
A uniform ladder $AB$, of mass $m$ and length $2a$, has one end $A$ on rough horizontal ground. The other end $B$ rests against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall. The ladder makes an angle $\alpha$ with the horizontal, where $\tan \alpha = \frac{3}{4}$. A child of mass $2m$ stands on the ladder at $C$ where $AC = \frac{1}{2}a$, as shown in Fig. 1. The ladder and the child are in equilibrium.

By modelling the ladder as a rod and the child as a particle, calculate the least possible value of the coefficient of friction between the ladder and the ground.

(9)

Jan 2003, Q3
A uniform steel girder $AB$, of mass 40 kg and length 3 m, is freely hinged at $A$ to a vertical wall. The girder is supported in a horizontal position by a steel cable attached to the girder at $B$. The other end of the cable is attached to the point $C$ vertically above $A$ on the wall, with $\angle ABC = \alpha$, where $\tan \alpha = \frac{3}{4}$. A load of mass 60 kg is suspended by another cable from the girder at the point $D$, where $AD = 2$ m, as shown in Fig. 2. The girder remains horizontal and in equilibrium. The girder is modelled as a rod, and the cables as light inextensible strings.

(a) Show that the tension in the cable $BC$ is 980 N.  

(b) Find the magnitude of the reaction on the girder at $A$.  

(c) Explain how you have used the modelling assumption that the cable at $D$ is light.  

June 2003, Q4
A uniform ladder, of weight $W$ and length $2a$, rests in equilibrium with one end $A$ on a smooth horizontal floor and the other end $B$ on a rough vertical wall. The ladder is in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the ladder is $\mu$. The ladder makes an angle $\theta$ with the floor, where $\tan \theta = 2$. A horizontal light inextensible string $CD$ is attached to the ladder at the point $C$, where $AC = \frac{1}{2}a$. The string is attached to the wall at the point $D$, with $BD$ vertical, as shown in Fig. 2. The tension in the string is $\frac{1}{4}W$. By modelling the ladder as a rod,

(a) find the magnitude of the force of the floor on the ladder, \hspace{1cm} (5)

(b) show that $\mu \geq \frac{1}{2}$. \hspace{1cm} (4)

(c) State how you have used the modelling assumption that the ladder is a rod. \hspace{1cm} (1)

---

8. A uniform ladder $AB$, of mass $m$ and length $2a$, has one end $A$ on rough horizontal ground. The coefficient of friction between the ladder and the ground is 0.6. The other end $B$ of the ladder rests against a smooth vertical wall.
A builder of mass 10\textit{m} stands at the top of the ladder. To prevent the ladder from slipping, the builder’s friend pushes the bottom of the ladder horizontally towards the wall with a force of magnitude \(P\). This force acts in a direction perpendicular to the wall. The ladder rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle \(\alpha\) with the horizontal, where 
\[
\tan \alpha = \frac{3}{5}.
\]

\(a\) Show that the reaction of the wall on the ladder has magnitude 7\textit{mg}.

\(b\) Find, in terms of \textit{m} and \textit{g}, the range of values of \(P\) for which the ladder remains in equilibrium.

---

A uniform rod \(AB\), of length 8\(a\) and weight \(W\), is free to rotate in a vertical plane about a smooth pivot at \(A\). One end of a light inextensible string is attached to \(B\). The other end is attached to point \(C\) which is vertically above \(A\), with \(AC = 6a\). The rod is in equilibrium with \(AB\) horizontal, as shown in Figure 1.

\(a\) By taking moments about \(A\), or otherwise, show that the tension in the string is \(\frac{5}{8}W\).

\(b\) Calculate the magnitude of the horizontal component of the force exerted by the pivot on the rod.

---

**Figure 1**

---

June 2004, Q6

---

Jan 2005, Q1
A uniform pole $AB$, of mass 30 kg and length 3 m, is smoothly hinged to a vertical wall at one end $A$. The pole is held in equilibrium in a horizontal position by a light rod $CD$. One end $C$ of the rod is fixed to the wall vertically below $A$. The other end $D$ is freely jointed to the pole so that $\angle ACD = 30^\circ$ and $AD = 0.5$ m, as shown in Figure 2. Find

(a) the thrust in the rod $CD$, \hspace{1cm} (4)

(b) the magnitude of the force exerted by the wall on the pole at $A$. \hspace{1cm} (6)

The rod $CD$ is removed and replaced by a longer light rod $CM$, where $M$ is the mid-point of $AB$. The rod is freely jointed to the pole at $M$. The pole $AB$ remains in equilibrium in a horizontal position.

(c) Show that the force exerted by the wall on the pole at $A$ now acts horizontally. \hspace{1cm} (2)

June 2005, Q6
A ladder $AB$, of weight $W$ and length $4a$, has one end $A$ on rough horizontal ground. The coefficient of friction between the ladder and the ground is $\mu$. The other end $B$ rests against a smooth vertical wall. The ladder makes an angle $\theta$ with the horizontal, where $\tan \theta = 2$. A load of weight $4W$ is placed at the point $C$ on the ladder, where $AC = 3a$, as shown in Figure 2. The ladder is modelled as a uniform rod which is in a vertical plane perpendicular to the wall. The load is modelled as a particle. Given that the system is in limiting equilibrium,

(a) show that $\mu = 0.35$.  

(6)

A second load of weight $kW$ is now placed on the ladder at $A$. The load of weight $4W$ is removed from $C$ and placed on the ladder at $B$. The ladder is modelled as a uniform rod which is in a vertical plane perpendicular to the wall. The loads are modelled as particles. Given that the ladder and the loads are in equilibrium,

(b) find the range of possible values of $k$.  

(7)

Jan 2006, Q6
A wooden plank $AB$ has mass $4m$ and length $4a$. The end $A$ of the plank lies on rough horizontal ground. A small stone of mass $m$ is attached to the plank at $B$. The plank is resting on a small smooth horizontal peg $C$, where $BC = a$, as shown in Figure 2. The plank is in equilibrium making an angle $\alpha$ with the horizontal, where $\tan \alpha = \frac{4}{3}$. The coefficient of friction between the plank and the ground is $\mu$. The plank is modelled as a uniform rod lying in a vertical plane perpendicular to the peg, and the stone as a particle. Show that

(a) the reaction of the peg on the plank has magnitude $\frac{16}{5} mg$,

(b) $\mu \geq \frac{48}{61}$.

(c) State how you have used the information that the peg is smooth.

June 2006, Q6

13.

A horizontal uniform rod $AB$ has mass $m$ and length $4a$. The end $A$ rests against a rough vertical wall. A particle of mass $2m$ is attached to the rod at the point $C$, where $AC = 3a$.
One end of a light inextensible string $BD$ is attached to the rod at $B$ and the other end is attached to the wall at a point $D$, where $D$ is vertically above $A$. The rod is in equilibrium in a vertical plane perpendicular to the wall. The string is inclined at an angle $\theta$ to the horizontal, where $\tan \theta = \frac{4}{3}$, as shown in Figure 2.

(a) Find the tension in the string.  

(b) Show that the horizontal component of the force exerted by the wall on the rod has magnitude $\frac{8}{3}mg$.  

The coefficient of friction between the wall and the rod is $\mu$. Given that the rod is in limiting equilibrium,

(c) find the value of $\mu$.  

---

14. Figure 3

A uniform beam $AB$ of mass 2 kg is freely hinged at one end $A$ to a vertical wall. The beam is held in equilibrium in a horizontal position by a rope which is attached to a point $C$ on the beam, where $AC = 0.14$ m. The rope is attached to the point $D$ on the wall vertically above $A$, where $\angle ACD = 30^\circ$, as shown in Figure 3. The beam is modelled as a uniform rod and the rope as a light inextensible string. The tension in the rope is 63 N.

Find

(a) the length of $AB$,  

(b) the magnitude of the resultant reaction of the hinge on the beam at $A$.  

---

Jan 2007, Q5

June 2007, Q5
A ladder $AB$, of mass $m$ and length $4a$, has one end $A$ resting on rough horizontal ground. The other end $B$ rests against a smooth vertical wall. A load of mass $3m$ is fixed on the ladder at the point $C$, where $AC = a$. The ladder is modelled as a uniform rod in a vertical plane perpendicular to the wall and the load is modelled as a particle. The ladder rests in limiting equilibrium making an angle of $30^\circ$ with the wall, as shown in Figure 2.

Find the coefficient of friction between the ladder and the ground.

(10)

Jan 2008, Q5

16.

A plank rests in equilibrium against a fixed horizontal pole. The plank is modelled as a uniform rod $AB$ and the pole as a smooth horizontal peg perpendicular to the vertical plane containing $AB$. The rod has length $3a$ and weight $W$ and rests on the peg at $C$,
where \( AC = 2a \). The end \( A \) of the rod rests on rough horizontal ground and \( AB \) makes an angle \( \alpha \) with the ground, as shown in Figure 2.

(a) Show that the normal reaction on the rod at \( A \) is \( \frac{1}{4} (4 - 3 \cos^2 \alpha) W \).

Given that the rod is in limiting equilibrium and that \( \cos \alpha = \frac{2}{3} \),

(b) find the coefficient of friction between the rod and the ground.

May 2008, Q5

17.

Figure 1

Figure 1 shows a ladder \( AB \), of mass 25 kg and length 4 m, resting in equilibrium with one end \( A \) on rough horizontal ground and the other end \( B \) against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall. The coefficient of friction between the ladder and the ground is \( \frac{1}{11} \). The ladder makes an angle \( \beta \) with the ground. When Reece, who has mass 75 kg, stands at the point \( C \) on the ladder, where \( AC = 2.8 \) m, the ladder is on the point of slipping. The ladder is modelled as a uniform rod and Reece is modelled as a particle.

(a) Find the magnitude of the frictional force of the ground on the ladder.

(b) Find, to the nearest degree, the value of \( \beta \).

(c) State how you have used the modelling assumption that Reece is a particle.

Jan 2009, Q2
A uniform rod $AB$, of length 1.5 m and mass 3 kg, is smoothly hinged to a vertical wall at $A$. The rod is held in equilibrium in a horizontal position by a light strut $CD$ as shown in Figure 1. The rod and the strut lie in the same vertical plane, which is perpendicular to the wall. The end $C$ of the strut is freely jointed to the wall at a point 0.5 m vertically below $A$. The end $D$ is freely joined to the rod so that $AD$ is 0.5 m.

(a) Find the thrust in $CD$.

(b) Find the magnitude and direction of the force exerted on the rod $AB$ at $A$.

May 2009, Q4
Figure 2 shows a uniform rod $AB$ of mass $m$ and length $4a$. The end $A$ of the rod is freely hinged to a point on a vertical wall. A particle of mass $m$ is attached to the rod at $B$. One end of a light inextensible string is attached to the rod at $C$, where $AC = 3a$. The other end of the string is attached to the wall at $D$, where $AD = 2a$ and $D$ is vertically above $A$. The rod rests horizontally in equilibrium in a vertical plane perpendicular to the wall and the tension in the string is $T$.

(a) Show that $T = mg\sqrt{13}$.

The particle of mass $m$ at $B$ is removed from the rod and replaced by a particle of mass $M$ which is attached to the rod at $B$. The string breaks if the tension exceeds $2mg\sqrt{13}$. Given that the string does not break,

(b) show that $M \leq \frac{5}{2}m$.

(a) find the work done in dragging the box from $A$ to $B$.

The box is released from rest at the point $B$ and slides down the slope. Using the work-energy principle, or otherwise,

(b) find the speed of the box as it reaches $A$.

A box of mass 30 kg is held at rest at point $A$ on a rough inclined plane. The plane is inclined at $20^\circ$ to the horizontal. Point $B$ is 50 m from $A$ up a line of greatest slope of the plane, as shown in Figure 1. The box is dragged from $A$ to $B$ by a force acting parallel to $AB$ and then held at rest at $B$. The coefficient of friction between the box and the plane is $\frac{1}{4}$. Friction is the only non-gravitational resistive force acting on the box. Modelling the box as a particle,

(a) find the work done in dragging the box from $A$ to $B$.

(b) find the speed of the box as it reaches $A$.
A uniform plank $AB$, of weight 100 N and length 4 m, rests in equilibrium with the end $A$ on rough horizontal ground. The plank rests on a smooth cylindrical drum. The drum is fixed to the ground and cannot move. The point of contact between the plank and the drum is $C$, where $AC = 3$ m, as shown in Figure 4. The plank is resting in a vertical plane which is perpendicular to the axis of the drum, at an angle $\alpha$ to the horizontal, where $\sin \alpha = \frac{3}{5}$. The coefficient of friction between the plank and the ground is $\mu$.

Modelling the plank as a rod, find the least possible value of $\mu$.

(10)
Jan 2011, Q7

A uniform rod $AB$, of mass $3m$ and length $4a$, is held in a horizontal position with the end $A$ against a rough vertical wall. One end of a light inextensible string $BD$ is attached to the rod at $B$ and the other end of the string is attached to the wall at the point $D$ vertically above $A$, where $AD = 3a$. A particle of mass $3m$ is attached to the rod at $C$, where $AC = x$. 

Figure 3

Figure 4
The rod is in equilibrium in a vertical plane perpendicular to the wall as shown in Figure 3. The tension in the string is \( \frac{25}{4} mg \).

Show that

(a) \( x = 3a \),

(b) the horizontal component of the force exerted by the wall on the rod has magnitude \( 5mg \).

The coefficient of friction between the wall and the rod is \( \mu \). Given that the rod is about to slip,

(c) find the value of \( \mu \).

A uniform rod \( AB \) has mass 4 kg and length 1.4 m. The end \( A \) is resting on rough horizontal ground. A light string \( BC \) has one end attached to \( B \) and the other end attached to a fixed point \( C \). The string is perpendicular to the rod and lies in the same vertical plane as the rod. The rod is in equilibrium, inclined at \( 20^\circ \) to the ground, as shown in Figure 2.

(a) Find the tension in the string.

Given that the rod is about to slip,

(b) find the coefficient of friction between the rod and the ground.
24.

A uniform rod $AB$, of mass 5 kg and length 4 m, has its end $A$ smoothly hinged at a fixed point. The rod is held in equilibrium at an angle of 25° above the horizontal by a force of magnitude $F$ newtons applied to its end $B$. The force acts in the vertical plane containing the rod and in a direction which makes an angle of 40° with the rod, as shown in Figure 1.

(a) Find the value of $F$.

(b) Find the magnitude and direction of the vertical component of the force acting on the rod at $A$.

25.

A ladder, of length 5 m and mass 18 kg, has one end $A$ resting on rough horizontal ground and its other end $B$ resting against a smooth vertical wall. The ladder lies in a vertical plane perpendicular to the wall and makes an angle $\alpha$ with the horizontal ground, where $\tan \alpha = \frac{3}{4}$, as shown in Figure 1. The coefficient of friction between the ladder and the ground is $\mu$. A woman of mass 60 kg stands on the ladder at the point $C$, where $AC =$
3 m. The ladder is on the point of slipping. The ladder is modelled as a uniform rod and
the woman as a particle.

Find the value of $\mu$.

(a) Show that $F = \frac{3amg \cos \theta}{b}$.

(b) Find, in terms of $a$, $b$, $g$, $m$ and $\theta$,

(i) the horizontal component of the force acting on the rod at $A$,

(ii) the vertical component of the force acting on the rod at $A$.

Given that the force acting on the rod at $A$ acts along the rod,

(c) find the value of $\frac{a}{b}$.
27. A rough circular cylinder of radius $4a$ is fixed to a rough horizontal plane with its axis horizontal. A uniform rod $AB$, of weight $W$ and length $6a\sqrt{3}$, rests with its lower end $A$ on the plane and a point $C$ of the rod against the cylinder. The vertical plane through the rod is perpendicular to the axis of the cylinder. The rod is inclined at $60^\circ$ to the horizontal, as shown in Figure 1.

![Figure 1](image)

(a) Show that $AC = 4a\sqrt{3}$.

(b) The coefficient of friction between the rod and the cylinder is $\frac{\sqrt{3}}{3}$ and the coefficient of friction between the rod and the plane is $\mu$. Given that friction is limiting at both $A$ and $C$,

(2) find the value of $\mu$.

June 2013_R, Q4
A non-uniform rod, \( AB \), of mass \( m \) and length \( 2l \), rests in equilibrium with one end \( A \) on a rough horizontal floor and the other end \( B \) against a rough vertical wall. The rod is in a vertical plane perpendicular to the wall and makes an angle of 60° with the floor as shown in Figure 1. The coefficient of friction between the rod and the floor is \( \frac{1}{4} \) and the coefficient of friction between the rod and the wall is \( \frac{2}{3} \). The rod is on the point of slipping at both ends.

(a) Find the magnitude of the vertical component of the force exerted on the rod by the floor.

(b) Find the distance \( AG \).

Figure 1

June 2014_R, Q3
A uniform rod \( AB \) of weight \( W \) has its end \( A \) freely hinged to a point on a fixed vertical wall. The rod is held in equilibrium, at angle \( \theta \) to the horizontal, by a force of magnitude \( P \). The force acts perpendicular to the rod at \( B \) and in the same vertical plane as the rod, as shown in Figure 3. The rod is in a vertical plane perpendicular to the wall. The magnitude of the vertical component of the force exerted on the rod by the wall at \( A \) is \( Y \).

(a) Show that \( Y = \frac{W}{2} \left( 2 - \cos^2 \theta \right) \).

(b) Given that \( \theta = 45^\circ \), find the magnitude of the force exerted on the rod by the wall at \( A \), giving your answer in terms of \( W \).

June 2014, Q7

30. A ladder \( AB \), of weight \( W \) and length \( 2l \), has one end \( A \) resting on rough horizontal ground. The other end \( B \) rests against a rough vertical wall. The coefficient of friction between the ladder and the wall is \( \frac{1}{3} \). The coefficient of friction between the ladder and the ground is \( \mu \). Friction is limiting at both \( A \) and \( B \). The ladder is at an angle \( \theta \) to the ground, where \( \tan \theta = \frac{5}{3} \). The ladder is modelled as a uniform rod which lies in a vertical plane perpendicular to the wall.

Find the value of \( \mu \).

June 2015, Q4
A non-uniform rod $AB$, of mass 5 kg and length 4 m, rests with one end $A$ on rough horizontal ground. The centre of mass of the rod is $d$ metres from $A$. The rod is held in limiting equilibrium at an angle $\theta$ to the horizontal by a force $P$, which acts in a direction perpendicular to the rod at $B$, as shown in Figure 2. The line of action of $P$ lies in the same vertical plane as the rod.

(a) Find, in terms of $d$, $g$ and $\theta$,

(i) the magnitude of the vertical component of the force exerted on the rod by the ground,

(ii) the magnitude of the friction force acting on the rod at $A$.

Given that $\tan \theta = \frac{5}{12}$ and that the coefficient of friction between the rod and the ground is $\frac{1}{2}$,

(b) find the value of $d$.

June 2016, Q5
Figure 2 shows a uniform rod $AB$, of mass $m$ and length $2a$, with the end $B$ resting on rough horizontal ground. The rod is held in equilibrium at an angle $\theta$ to the vertical by a light inextensible string. One end of the string is attached to the rod at the point $C$, where $AC = \frac{2}{3}a$. The other end of the string is attached to the point $D$, which is vertically above $B$, where $BD = 2a$.

(a) By taking moments about $D$, show that the magnitude of the frictional force acting on the rod at $B$ is $\frac{1}{2}mg \sin \theta$.

(b) Find the magnitude of the normal reaction on the rod at $B$.

The rod is in limiting equilibrium when $\tan \theta = \frac{4}{3}$.

(c) Find the coefficient of friction between the rod and the ground.
A uniform rod $AB$ of weight $W$ is freely hinged at end $A$ to a vertical wall. The rod is supported in equilibrium at an angle of $60^\circ$ to the wall by a light rigid strut $CD$. The strut is freely hinged to the rod at the point $D$ and to the wall at the point $C$, which is vertically below $A$, as shown in Figure 1. The rod and the strut lie in the same vertical plane, which is perpendicular to the wall. The length of the rod is $4a$ and $AC = AD = 2.5a$.

(a) Show that the magnitude of the thrust in the strut is $\frac{4\sqrt{3}}{5}W$.

(b) Find the magnitude of the force acting on the rod at $A$. 

June 2014, IAL, Q3
A uniform rod $AB$, of mass $m$ and length $2a$, is freely hinged to a fixed point $A$. A particle of mass $km$ is fixed to the rod at $B$. The rod is held in equilibrium, at an angle $\theta$ to the horizontal, by a force of magnitude $F$ acting at the point $C$ on the rod, where $AC = \frac{5}{4}a$, as shown in Figure 2. The line of action of the force at $C$ is at right angles to $AB$ and in the vertical plane containing $AB$.

Given that $\tan \theta = \frac{3}{4}$

(a) show that $F = \frac{16}{25} mg(1 + 2k)$, 

(4)

(b) find, in terms of $m$, $g$ and $k$,

(i) the horizontal component of the force exerted by the hinge on the rod at $A$,

(ii) the vertical component of the force exerted by the hinge on the rod at $A$.

(5)

Given also that the force acting on the rod at $A$ acts at $45^\circ$ above the horizontal,

(c) find the value of $k$.

(3)

Jan 2015, IAL, Q5
A uniform rod $AB$ has length $4a$ and weight $W$. A particle of weight $kW$, $k < 1$, is attached to the rod at $B$. The rod rests in equilibrium against a fixed smooth horizontal peg. The end $A$ of the rod is on rough horizontal ground, as shown in Figure 2. The rod rests on the peg at $C$, where $AC = 3a$, and makes an angle $\alpha$ with the ground, where $\tan \alpha = \frac{1}{3}$. The peg is perpendicular to the vertical plane containing $AB$.

(a) Give a reason why the force acting on the rod at $C$ is perpendicular to the rod. 

(b) Show that the magnitude of the force acting on the rod at $C$ is

$$\frac{\sqrt{10}}{5} W (1 + 2k)$$

The coefficient of friction between the rod and the ground is $\frac{3}{4}$.

(c) Show that for the rod to remain in equilibrium $k \leq \frac{2}{11}$.
A uniform rod $AB$, of mass $3m$ and length $2a$, is freely hinged at $A$ to a fixed point on horizontal ground. A particle of mass $m$ is attached to the rod at the end $B$. The system is held in equilibrium by a force $F$ acting at the point $C$, where $AC = b$. The rod makes an acute angle $\theta$ with the ground, as shown in Figure 3. The line of action of $F$ is perpendicular to the rod and in the same vertical plane as the rod.

(a) Show that the magnitude of $F$ is $\frac{5mga}{b} \cos \theta$. 

(4)

The force exerted on the rod by the hinge at $A$ is $R$, which acts upwards at an angle $\phi$ above the horizontal, where $\phi > \theta$.

(b) Find

(i) the component of $R$ parallel to the rod, in terms of $m$, $g$ and $\theta$,

(ii) the component of $R$ perpendicular to the rod, in terms of $a$, $b$, $m$, $g$ and $\theta$. 

(5)

(c) Hence, or otherwise, find the range of possible values of $b$, giving your answer in terms of $a$. 

(2)
A uniform rod $AB$ has mass 6 kg and length 2 m. The end $A$ of the rod rests against a rough vertical wall. One end of a light string is attached to the rod at $B$. The other end of the string is attached to the wall at $C$, which is vertically above $A$. The angle between the rod and the string is $30^\circ$ and the angle between the rod and the wall is $70^\circ$, as shown in Figure 3. The rod is in a vertical plane perpendicular to the wall and rests in limiting equilibrium.

Find

(a) the tension in the string, 

(b) the coefficient of friction between the rod and the wall,

(c) the direction of the force exerted on the rod by the wall at $A$. 

June 2016, IAL, Q5
A uniform rod $AB$ of length 8 m and weight $W$ newtons rests in equilibrium against a rough horizontal peg $P$. The end $A$ is on rough horizontal ground. The friction is limiting at both $A$ and $P$. The distance $AP$ is 5 m, as shown in Figure 1. The rod rests at angle $\theta$ to the horizontal, where $\tan \theta = \frac{4}{3}$. The rod is in a vertical plane which is perpendicular to $P$.

The coefficient of friction between the rod and $P$ is $\frac{1}{4}$ and the coefficient of friction between the rod and the ground is $\mu$.

(a) Show that the magnitude of the normal reaction between the rod and $P$ is $0.48W$ newtons. 

(b) Find the value of $\mu$. 

Oct 2016, IAL, Q5
39.

A uniform rod $AB$ has mass $m$ and length $2a$. The end $A$ is in contact with rough horizontal ground and the end $B$ is in contact with a smooth vertical wall. The rod rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle of $30^\circ$ with the wall, as shown in Figure 2. The coefficient of friction between the rod and the ground is $\mu$.

(a) Find, in terms of $m$ and $g$, the magnitude of the force exerted on the rod by the wall. (4)

(b) Show that $\mu \geq \frac{\sqrt{3}}{6}$ (3)

A particle of mass $km$ is now attached to the rod at $B$. Given that $\mu = \frac{\sqrt{3}}{5}$ and that the rod is now in limiting equilibrium,

(c) find the value of $k$. (6)

Jan 2017, IAL, Q7