Differentiation C3

Specifications.

By the end of this unit you should be able to:

Use chain rule to find the derivative of composite functions.

Find the derivative of products and quotients.

Find the derivatives of trigonometric, logarithmic and exponential functions.

Work will also include turning points and the equations of tangents and normals.
The following differentials must be learnt.

<table>
<thead>
<tr>
<th>Function</th>
<th>Differential</th>
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<tbody>
<tr>
<td>( \ln x )</td>
<td>( \frac{1}{x} )</td>
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<tr>
<td>( e^{mx} )</td>
<td>( me^{mx} )</td>
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<tr>
<td>( \sin t )</td>
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<td>( \cos t )</td>
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<td>( \tan t )</td>
<td>( \sec^2 t )</td>
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<td>( \sec t )</td>
<td>( \sec t \tan t )</td>
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<td>( \cot t )</td>
<td>( -\csc^2 t )</td>
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<td>( \csc t )</td>
<td>( -\csc t \cot t )</td>
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<tr>
<td>( \sin^n t )</td>
<td>( n\sin^{n-1} t \cos t )</td>
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<tr>
<td>( \cos^n t )</td>
<td>( -n\cos^{n-1} t \sin t )</td>
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<td>( \sin (g(t)) )</td>
<td>( g'(t)\cos (g(t)) )</td>
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<td>( \cos (g(t)) )</td>
<td>( -g'(t)\sin (g(t)) )</td>
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<tr>
<td>( \tan (g(t)) )</td>
<td>( g'(t)\sec^2 (g(t)) )</td>
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**Chain Rule**

If \( y \) is a function of \( v \) and \( v \), in turn, is a function of \( x \), then:

\[
\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx}
\]

**Product Rule**

\[
\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}
\]

**Quotient Rule**

\[
\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]
**Composite Functions**

\[ y = (mx + c)^n \]

\[ \frac{dy}{dx} = nm(mx + c)^{n-1} \]

\[ y = (f(x))^n \]

\[ \frac{dy}{dx} = nf'(x) \times (f(x))^{n-1} \]

The following example covers most of the ideas introduced above.

**Example 1**

Differentiate the following with respect to \( x \) simplify your answer as far as possible.

a) \((x + \ln 4x)^7\)

b) \(6 \sin^2x + \sec 2x\)

c) \(x^{11} \tan 13x\)

d) \(x^5 e^{5x + 5}\)

e) \(\frac{\cos 5x^5}{3x}\)
a) \[ \frac{d}{dx}(x + \ln 4x)^7 \]

This is a composite function and chain rule must be used.

By chain rule

If \( y = (x + \ln 4x)^7 \)  
let \( u = (x + \ln 4x) \)

So \( y = u^7 \) 
\[ \frac{du}{dx} = 1 + \frac{1}{x} \quad \text{not} \quad \frac{1}{4x} \]

\[ \frac{dy}{du} = 7u^6 \]

So

\[ \frac{d}{dx}(x + \ln 4x)^7 = 7u^6 \times \left(1 + \frac{1}{x}\right) \]

\[ = 7(x + \ln 4x)^6 \left(1 + \frac{1}{x}\right) \]

b) The differential of \( \sec 2x \) is \( 2 \sec 2x \tan 2x \) (don't forget the 2's). The \( \sin^2 x \) requires substitution and the use of chain rule.

If \( y = 6\sin^2 x \)  
let \( u = \sin x \)

then \( y = 6u^2 \) 
\[ \frac{du}{dx} = \cos x \]

\[ \frac{dy}{du} = 12u \]
By chain rule

\[ \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \]

\[ = \cos x \times 12u \]

\[ = 12\cos x \sin x = 6 \sin 2x \]

So finally:

\[ \frac{d}{dx}(6\sin^2 x + \sec 2x) = 6 \sin 2x + 2 \sec 2x \tan 2x \]

c) The rule for differentiating products is to differentiate the first and times it by the second and then add the differential of the second times by the first.

Or in symbols:

\[ \frac{d(\text{uv})}{dx} = v \frac{du}{dx} + u \frac{dv}{dx} \]

\[ \frac{d}{dx}(x^{11} \tan 13x) = 11x^{10} \tan 13x + 13x^{11} \sec^2 13x \]

A lot of students forget to put the 13 back into the trigonometric function and write \( \sec^2 x \).

d) \[ \frac{d}{dx}(x^5 e^{5x+5}) \]

Another product to differentiate, remember to differentiate the power on the e and bring it to the front.

\[ \frac{d}{dx}(x^5 e^{5x+5}) = 5x^4 e^{5x+5} + 5x^5 e^{5x+5} \]

\[ = 5x^4 e^{5x+5}(1+x) \]
e) \( \frac{d}{dx} \left( \frac{\cos5x^5}{3x} \right) \)

We are asked to differentiate a quotient but this can be rewritten as a product. Most questions can be treated this way unless a question says specifically to use the quotient rule.

\[
\frac{d}{dx} \left( \frac{\cos5x^5}{3x} \right) = \frac{d}{dx} \left( \frac{x^{-1} \cos5x^5}{3} \right)
\]

Let \( u = \frac{x^{-1}}{3} \) \quad v = \cos5x^5

\[
\frac{du}{dx} = \frac{-x^{-2}}{3} \quad \frac{dv}{dx} = -25x^4 \sin5x^5
\]

Therefore:

\[
\frac{d}{dx} \left( \frac{\cos5x^5}{3x} \right) = \frac{-x^{-2}}{3} \cos5x^5 - \frac{x^{-1}}{3} \times -25x^4 \sin5x^5
\]

\[
= -\frac{\cos5x^5}{3x^2} + \frac{25x^3 \sin5x^5}{3}
\]

Example 2

Given that \( x = 8\sin(7y + 3) \), find \( \frac{dy}{dx} \).

The only thing that is new here is that \( \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \)

\[
x = 8\sin(7y + 3)
\]

Don’t forget to write the \((7y + 3)\) in

\[
\frac{dx}{dy} = 56\cos(7y + 3)
\]

\[
\frac{dy}{dx} = \frac{1}{56\cos(7y + 3)} = \frac{\sec(7y + 3)}{56}
\]
Equations of tangents and Normals

Example 3

The curve with equation \( y = \frac{1}{6} e^x \) meets the y axis at the point A.

a) Prove that the tangent at A to the curve has equation 6y = x + 1

b) The point B has x-coordinate \( \ln 36 \) and lies on the curve. The normal at B to the curve meets the tangent at A to the curve at the point C. Prove that the x coordinate of C is \( \ln 6 + 17.5 \) and find the y coordinate of C.

It is always best practice to make a sketch of the diagram especially in this case where there are quite a few lines.

\[
\frac{dy}{dx} = \frac{1}{6} e^x
\]

At A, \( x = 0 \) therefore the gradient is \( \frac{1}{6} \) as is the y coordinate.

\[
y = mx + c
\]
\[
\frac{1}{6} = \frac{1}{6} \times 0 + c
\]

Therefore
\[
y = \frac{1}{6}x + \frac{1}{6}
\]
and hence
\[
6y = x + 1
\]

b) Find the equation of the normal at the point B with x coord of ln 36.

From part (a) the gradient function is \( \frac{1}{6}e^x \) hence the gradient at B is 6 and so the normal gradient is \(-\frac{1}{6}\). (y coord also 6)

\[
y = mx + c
\]
\[
6 = \frac{-\ln 36}{6} + c
\]
\[
y = \frac{-x}{6} + 6 + \frac{\ln 36}{6}
\]

The normal then meets the tangent from part (a) at the point C.

\[
\frac{1}{6}x + \frac{1}{6} = \frac{-x}{6} + 6 + \frac{\ln 36}{6}
\]
\[
2x = 35 + \ln 36
\]
\[
x = 17.5 + \frac{1}{2}\ln 36 = 17.5 + \ln 6
\]

Therefore the y coordinate can be found by substituting \( x = \ln 6 + 17.5 \) into \( 6y = x + 1 \).

\[
6y = \ln 6 + 17.5 + 1
\]
\[
y = \frac{1}{6} \left( \ln 6 + \frac{37}{2} \right)
\]
Example 4

The curve $c$ has equation $y = 4x^{\frac{7}{2}} - \ln4x$, where $x > 0$. The tangent at the point $C$ where $x = 1$ meets the $x$ axis at the point $A$.

Prove that the $x$ coordinate of $A$ is $\frac{9 + \ln4}{13}$.

Once again start by differentiating to find the gradient when $x = 1$.

\[ y = 4x^{\frac{7}{2}} - \ln4x \]

\[ \frac{dy}{dx} = 14x^{\frac{5}{2}} - \frac{1}{x} \]

When $x = 1$  grad = 13  $y = 4 - \ln4$

$y = mx + c$

$4 - \ln4 = 13 + c$

$c = -9 - \ln4$  \hspace{1cm}  $y = 13x - 9 - \ln4$

The line meets the $x$-axis at the point where $y = 0$. Therefore:

$13x - 9 - \ln4 = 0$

$x = \frac{9 + \ln4}{13}$
Differentiating Quotients

Example 5

Given that $y = \frac{4x^2 - 16x + 7}{(x - 2)^2}$, $x \neq 2$,

Show that $\frac{dy}{dx} = \frac{18}{(x - 2)^3}$

It is obvious from the question that by using the quotient rule it will be easier to get the desired answer.

$$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$u = 4x^2 - 16x + 7$ \hspace{1cm} $v = (x - 2)^2$

$$\frac{du}{dx} = 8x - 16$$ \hspace{1cm} $$\frac{dv}{dx} = 2(x - 2)$$

Therefore:
\[
\frac{dy}{dx} = \frac{(x - 2)^2(8x - 16) - (4x^2 - 16x + 7) \times 2(x - 2)}{(x - 2)^4}
\]
cancel a factor of \((x - 2)\) top and bottom

\[
= \frac{(x - 2)(8x - 16) - 8x^2 + 32x - 14}{(x - 2)^3}
\]

\[
= \frac{8x^2 - 32x + 32 - 8x^2 + 32x - 14}{(x - 2)^3}
\]

\[
= \frac{18}{(x - 2)^3}
\]

Example 6

The diagram shows part of the curve with equation,

\[y = (10x - 3) \tan 3x, \quad 0 \leq x < \frac{\pi}{4}\]

The curve has a minimum at the point \(P\). The \(x\) coordinate of \(P\) is \(K\). Show that \(K\) satisfies the equation \(30K - 9 + 5 \sin 6K = 0\)
As soon as you see the word minimum your first thought should be to differentiate and set it equal to zero.

Differentiating a product:

\[ y = (10x - 3) \tan 3x \]

\[ \frac{dy}{dx} = 10 \tan 3x + 3(10x - 3) \sec^2 3x \]

At a minimum the gradient is zero

\[ 0 = 10 \tan 3x + 3(10x - 3) \sec^2 3x \]

\[ 0 = 10 \frac{\sin 3x}{\cos 3x} + \frac{3(10x - 3)}{\cos^2 3x} \]

Multiply by \( \cos^2 3x \)

\[ 0 = 10 \sin 3x \cos 3x + 3(10x - 3) \]

\[ 0 = 5 \sin 6x + 3(10x - 3) \]

If \( x = K \)

\[ 30K - 9 + 5 \sin 6K = 0 \]

A lot of the ideas outlined above are not complicated and the final example below deals with turning points and the differential of exponential functions.
Example 7

a) The curve, \( C \), has equation

\[
y = \frac{x}{9 + x^2}
\]

Use calculus to find the coordinates of the turning points of \( C \).

b) Given that

\[
y = \left(1 + e^{4x}\right)^{\frac{5}{4}}
\]

find the value of \( \frac{dy}{dx} \) at the point \( x = \frac{1}{4} \ln 3 \).

a) The curve \( C \) is to be differentiated as a quotient

\[
\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]

\[
u = x \quad v = 9 + x^2
\]

\[
\frac{du}{dx} = 1 \quad \frac{dv}{dx} = 2x
\]

\[
\frac{dy}{dx} = \frac{9 + x^2 - 2x^2}{(9 + x^2)}
\]

\[
\frac{dy}{dx} = \frac{9 - x^2}{(9 + x^2)}
\]
Turning points exist where \( \frac{dy}{dx} = 0 \) therefore the numerator must equal zero (why not the denominator?). The numerator is the difference of two squares and therefore the values of \( x \) must be +/- 3.

By substituting these values into \( y \) we get:

\[
y = \frac{x}{9 + x^2}
\]

\[
\begin{array}{c}
(\frac{3}{6}, \frac{1}{6}) \\
(\frac{3}{6}, \frac{-1}{6})
\end{array}
\]

b) Given that

\[
y = (1 + e^{4x})^\frac{5}{4}
\]

find the value of \( \frac{dy}{dx} \) at the point \( x = \frac{1}{4} \ln 3 \).

This is a composite function. To differentiate it simply multiply by the power, multiply by the differential of the bracket and then multiply by the differential of the bracket.

\[
y = (1 + e^{4x})^\frac{5}{4}
\]

\[
\frac{dy}{dx} = \frac{5}{4} \times 4e^{4x} \times (1 + e^{4x})^{\frac{1}{4}}
\]

\[
\frac{dy}{dx} = 5e^{4x}(1 + e^{4x})^{\frac{1}{4}}
\]

Let \( x = \frac{1}{4} \ln 3 \)

\[
\frac{dy}{dx} = 5e^{\ln 3}(1 + e^{\ln 3})^{\frac{1}{4}}
\]

\[
\frac{dy}{dx} = 15\left(4^{\frac{1}{4}}\right)
\]

C3 Differentiation is an area where marks can be scored easily. You should not find these questions difficult!
Edexcel past examination questions

1. (a) Differentiate with respect to $x$
   (i) $3 \sin^2 x + \sec 2x$,
   (ii) $(x + \ln(2x))^3$.

   Given that $y = \frac{5x^2 - 10x + 9}{(x - 1)^2}$, $x \neq 1$,
   (b) show that $\frac{dy}{dx} = -\frac{8}{(x - 1)^3}$.

   [2005 June Q2]

2. The point $P$ lies on the curve with equation $y = \ln \left( \frac{1}{3}x \right)$. The $x$-coordinate of $P$ is 3.

   Find an equation of the normal to the curve at the point $P$ in the form $y = ax + b$, where $a$ and $b$ are constants.

   [2006 Jan Q3]

3. (a) Differentiate with respect to $x$
   (i) $x^2e^{3x^2}$,
   (ii) $\frac{\cos(2x^3)}{3x}$.

   (b) Given that $x = 4 \sin (2y + 6)$, find $\frac{dy}{dx}$ in terms of $x$.

   [2006 Jan Q4]

4. Differentiate, with respect to $x$,
   (a) $e^{3x} + \ln 2x$,
   (b) $(5 + x^2)^{\frac{3}{2}}$.

   [2006 June Q2]
The curve $C$ has equation $x = 2 \sin y$.

(a) Show that the point $P\left(\sqrt{2}, \frac{\pi}{4}\right)$ lies on $C$. 

(b) Show that $\frac{dy}{dx} = \frac{1}{\sqrt{2}}$ at $P$. 

(c) Find an equation of the normal to $C$ at $P$. Give your answer in the form $y = mx + c$, where $m$ and $c$ are exact constants.

(i) The curve $C$ has equation $y = \frac{x}{9 + x^2}$.

Use calculus to find the coordinates of the turning points of $C$. 

(ii) Given that $y = (1 + e^{2x})^{3/2}$, find the value of $\frac{dy}{dx}$ at $x = \frac{1}{2} \ln 3$.

A curve $C$ has equation $y = x^2 e^x$ 

(a) Find $\frac{dy}{dx}$, using the product rule for differentiation. 

(b) Hence find the coordinates of the turning points of $C$. 

(c) Find $\frac{d^2y}{dx^2}$. 

(d) Determine the nature of each turning point of the curve $C$. 

A curve $C$ has equation 

$$y = e^{2x} \tan x, \quad x \neq (2n + 1) \frac{\pi}{2}.$$ 

(a) Show that the turning points on $C$ occur where $\tan x = -1$. 

(b) Find an equation of the tangent to $C$ at the point where $x = 0$. 

The point $P$ lies on the curve with equation
\[ y = 4e^{2x} + 1. \]

The \( y \)-coordinate of \( P \) is 8.

(a) Find, in terms of ln 2, the \( x \)-coordinate of \( P \).  

(b) Find the equation of the tangent to the curve at the point \( P \) in the form \( y = ax + b \), where \( a \) and \( b \) are exact constants to be found.

10. (a) Differentiate with respect to \( x \),
    (i) \( e^{3x}(\sin x + 2 \cos x) \),
    (ii) \( x^3 \ln (5x + 2) \).

Given that \( y = \frac{3x^2 + 6x - 7}{(x+1)^2} \), \( x \neq -1 \),

(b) show that \( \frac{dy}{dx} = \frac{20}{(x+1)^3} \).

(c) Hence find \( \frac{d^2y}{dx^2} \) and the real values of \( x \) for which \( \frac{d^2y}{dx^2} = -\frac{15}{4} \).

11. (a) Find the value of \( \frac{dy}{dx} \) at the point where \( x = 2 \) on the curve with equation

\[ y = x^2 \sqrt{5x - 1}. \]

(b) Differentiate \( \frac{\sin 2x}{x^2} \) with respect to \( x \).

12. Find the equation of the tangent to the curve \( x = \cos (2y + \pi) \) at \( \left( 0, \frac{\pi}{4} \right) \).

Give your answer in the form \( y = ax + b \), where \( a \) and \( b \) are constants to be found.
13. \( f(x) = \frac{2x + 2}{x^2 - 2x - 3} - \frac{x + 1}{x - 3} \).

(a) Express \( f(x) \) as a single fraction in its simplest form. (4)

(b) Hence show that \( f'(x) = \frac{2}{(x - 3)^2} \). (3)

[2009 Jan Q2]

14. (i) Differentiate with respect to \( x \)

(a) \( x^2 \cos 3x \), (3)

(b) \( \frac{\ln(x^2 + 1)}{x^2 + 1} \). (4)

(ii) A curve \( C \) has the equation

\[ y = \sqrt{4x+1}, \quad x > -\frac{1}{4}, \quad y > 0. \]

The point \( P \) on the curve has \( x \)-coordinate 2. Find an equation of the tangent to \( C \) at \( P \) in the form \( ax + by + c = 0 \), where \( a, b \) and \( c \) are integers. (6)

[2009 June Q4]

15. (i) Given that \( y = \frac{\ln(x^2 + 1)}{x} \), find \( \frac{dy}{dx} \). (4)

(ii) Given that \( x = \tan y \), show that \( \frac{dy}{dx} = \frac{1}{1 + x^2} \). (5)

[2010 Jan Q4]

16. (a) By writing \( \sec x \) as \( \frac{1}{\cos x} \), show that \( \frac{d}{dx}(\sec x) = \sec x \tan x \). (3)

Given that \( y = e^{2x} \sec 3x \),

(b) \( \frac{dy}{dx} \). (4)

The curve with equation \( y = e^{2x} \sec 3x, \quad -\frac{\pi}{6} < x < \frac{\pi}{6} \), has a minimum turning point at \((a, b)\).

(c) Find the values of the constants \( a \) and \( b \), giving your answers to 3 significant figures. (4)

[2010 Jan Q7]
17. A curve $C$ has equation

$$y = \frac{3}{(5 - 3x)^2}, \quad x \neq \frac{5}{3}.$$ 

The point $P$ on $C$ has $x$-coordinate 2.
Find an equation of the normal to $C$ at $P$ in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are integers.  

[2010 June Q2]

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18.

Figure 1

Figure 1 shows a sketch of the curve $C$ with the equation $y = (2x^2 - 5x + 2)e^{-x}$.

(a) Find the coordinates of the point where $C$ crosses the $y$-axis.

(b) Show that $C$ crosses the $x$-axis at $x = 2$ and find the $x$-coordinate of the other point where $C$ crosses the $x$-axis.

(c) Find $\frac{dy}{dx}$. 

(d) Hence find the exact coordinates of the turning points of $C$. 

[2010 June Q5]
19. The curve $C$ has equation

$$y = \frac{3 + \sin 2x}{2 + \cos 2x}.$$  

(a) Show that

$$\frac{dy}{dx} = \frac{6\sin 2x + 4\cos 2x + 2}{(2 + \cos 2x)^2}$$  

(b) Find an equation of the tangent to $C$ at the point on $C$ where $x = \frac{\pi}{2}$.

Write your answer in the form $y = ax + b$, where $a$ and $b$ are exact constants.

[2011 Jan Q7]

20. Given that

$$\frac{d}{dx} (\cos x) = -\sin x,$$

(a) show that $\frac{d}{dx} (\sec x) = \sec x \tan x$.

Given that $x = \sec 2y$,

(b) find $\frac{dx}{dy}$ in terms of $y$.

(c) Hence find $\frac{dy}{dx}$ in terms of $x$.

[2010 Jan Q8]

21. Differentiate with respect to $x$

(a) $\ln (x^2 + 3x + 5)$,

(b) $\frac{\cos x}{x^2}$.

[2011 June Q1]
22. \[ f(x) = \frac{4x - 5}{(2x + 1)(x - 3)} - \frac{2x}{x^2 - 9}, \quad x \neq \pm 3, x \neq -\frac{1}{2}. \]

(a) Show that
\[ f(x) = \frac{5}{(2x + 1)(x - 3)}. \]

The curve \( C \) has equation \( y = f(x) \). The point \( P \left(-1, -\frac{5}{2}\right) \) lies on \( C \).

(b) Find an equation of the normal to \( C \) at \( P \).

23. Differentiate with respect to \( x \), giving your answer in its simplest form,

(a) \( x^2 \ln (3x) \),

(b) \( \frac{\sin 4x}{x^3} \).

24. The point \( P \) is the point on the curve \( x = 2 \tan \left(y + \frac{\pi}{12}\right) \) with \( y \)-coordinate \( \frac{\pi}{4} \).

Find an equation of the normal to the curve at \( P \).
25.

Figure 1 shows a sketch of the curve $C$ which has equation

$y = e^{x^3} \sin 3x, \quad \frac{-\pi}{3} \leq x \leq \frac{\pi}{3}$.

(a) Find the $x$-coordinate of the turning point $P$ on $C$, for which $x > 0$.
   Give your answer as a multiple of $\pi$.

(b) Find an equation of the normal to $C$ at the point where $x = 0$.

[2012 June Q3]

26. (a) Differentiate with respect to $x$,

(i) $x^2 \ln (3x)$,

(ii) $\frac{1-10x}{(2x-1)^5}$, giving your answer in its simplest form.

(b) Given that $x = 3 \tan 2y$ find $\frac{dy}{dx}$ in terms of $x$.
27. The curve $C$ has equation

$$y = (2x - 3)^5$$

The point $P$ lies on $C$ and has coordinates $(w, -32)$.

Find

(a) the value of $w$,

(b) the equation of the tangent to $C$ at the point $P$ in the form $y = mx + c$, where $m$ and $c$ are constants.

28. (i) Differentiate with respect to $x$

(a) $y = x^3 \ln 2x$,

(b) $y = (x + \sin 2x)^3$.

Given that $x = \cot y$,

(ii) show that $\frac{dy}{dx} = \frac{-1}{1 + x^2}$.
29. \( h(x) = \frac{2}{x+2} + \frac{4}{x^2 + 5} - \frac{18}{(x^2 + 5)(x+2)}, \quad x \geq 0. \)

(a) Show that \( h(x) = \frac{2x}{x^2 + 5} \). (4)

(b) Hence, or otherwise, find \( h'(x) \) in its simplest form. (3)

Figure 2

Figure 2 shows a graph of the curve with equation \( y = h(x) \).

(c) Calculate the range of \( h(x) \). (5)

30. Given that \( x = \sec^2 3y, \quad 0 < y < \frac{\pi}{6} \)

(a) find \( \frac{dx}{dy} \) in terms of \( y \). (2)

(b) Hence show that \( \frac{dy}{dx} = \frac{1}{6x(x-1)^2} \). (4)

(c) Find an expression for \( \frac{d^2y}{dx^2} \) in terms of \( x \). Give your answer in its simplest form. (4)

[2013 Jan Q7]

[2013 June Q5]
Figure 2 shows a sketch of part of the curve with equation \( y = f(x) \) where
\[
f(x) = (x^2 + 3x + 1)e^{x^2}
\]
The curve cuts the \( x \)-axis at points \( A \) and \( B \) as shown in Figure 2.

(a) Calculate the \( x \)-coordinate of \( A \) and the \( x \)-coordinate of \( B \), giving your answers to 3 decimal places.

(b) Find \( f'(x) \).

(c) Show that the \( x \)-coordinate of \( P \) is the solution of
\[
x = \frac{-3(2x^2 + 1)}{2(x^2 + 2)}
\]

The curve has a minimum turning point \( P \) as shown in Figure 2.

32. The curve \( C \) has equation \( y = f(x) \) where
\[
f(x) = \frac{4x + 1}{x - 2}, \quad x > 2
\]

(a) Show that
\[
f'(x) = \frac{-9}{(x - 2)^2}
\]
Given that \( P \) is a point on \( C \) such that \( f'(x) = -1 \),

(b) find the coordinates of \( P \).
33. The curve $C$ has equation $x = 8y \tan 2y$.

The point $P$ has coordinates $\left( \frac{\pi}{8}, \frac{\pi}{8} \right)$.

(a) Verify that $P$ lies on $C$.

(b) Find the equation of the tangent to $C$ at $P$ in the form $ay = x + b$, where the constants $a$ and $b$ are to be found in terms of $\pi$.

34. (i) Given that

$$x = \sec^2 2y, \quad 0 < y < \frac{\pi}{4}$$

show that

$$\frac{dy}{dx} = \frac{1}{4x\sqrt{(x-1)}}$$

(ii) Given that

$$y = \left( x^2 + x^3 \right) \ln 2x$$

find the exact value of $\frac{dy}{dx}$ at $x = \frac{e}{2}$, giving your answer in its simplest form.

(iii) Given that

$$f(x) = \frac{3\cos x}{(x+1)^\frac{5}{2}}, \quad x \neq -1$$

show that

$$f'(x) = \frac{g(x)}{(x+1)^\frac{5}{2}}, \quad x \neq -1$$

where $g(x)$ is an expression to be found.
35. The point $P$ lies on the curve with equation

$$x = (4y - \sin 2y)^2.$$ 

Given that $P$ has $(x, y)$ coordinates $\left(p, \frac{\pi}{2}\right)$, where $p$ is a constant,

(a) find the exact value of $p$. 

(1)

The tangent to the curve at $P$ cuts the $y$-axis at the point $A$.

(b) Use calculus to find the coordinates of $A$. 

[2015 June Q5]

36. Given that $k$ is a negative constant and that the function $f(x)$ is defined by

$$f(x) = 2 - \frac{(x-5k)(x-k)}{x^2 - 3kx + 2k^2}, \quad x \geq 0,$$

(a) show that $f(x) = \frac{x+k}{x-2k}$. 

(3)

(b) Hence find $f'(x)$, giving your answer in its simplest form. 

(3)

(c) State, with a reason, whether $f(x)$ is an increasing or a decreasing function. Justify your answer. 

(2)

[2015 June Q9]

37. 

$$y = \frac{4x}{x^2 + 5}.$$ 

(a) Find $\frac{dy}{dx}$, writing your answer as a single fraction in its simplest form. 

(4)

(b) Hence find the set of values of $x$ for which $\frac{dy}{dx} < 0$. 

(3)

[2016 June Q2]
38. (i) Find, using calculus, the $x$ coordinate of the turning point of the curve with equation

$$y = e^{3x} \cos 4x, \quad \frac{\pi}{4} \leq x < \frac{\pi}{2}.$$ 

Give your answer to 4 decimal places. (5)

(ii) Given $x = \sin^2 2y, \ 0 < y < \frac{\pi}{4}$, find $\frac{dy}{dx}$ as a function of $y$.

Write your answer in the form

$$\frac{dy}{dx} = p \csc(qy), \quad 0 < y < \frac{\pi}{4},$$

where $p$ and $q$ are constants to be determined. (5)

[2016 June Q5]

39. $f(x) = \frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6}, \quad x > 2, \quad x \in \mathbb{R}$.

(a) Given that

$$\frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6} \equiv x^2 + A + \frac{B}{x - 2},$$

find the values of the constants $A$ and $B$. (4)

(b) Hence or otherwise, using calculus, find an equation of the normal to the curve with equation $y = f(x)$ at the point where $x = 3$. (5)

[2016 June Q6]
40. (i) Given \( y = 2x(x^2 - 1)^5 \), show that
\[
\frac{dy}{dx} = g(x)(x^2 - 1)^4 \text{ where } g(x) \text{ is a function to be determined.}
\]
\[(4)\]

(b) Hence find the set of values of \( x \) for which \( \frac{dy}{dx} \geq 0 \)
\[(2)\]

(ii) Given
\[ x = \ln(\sec^2 y), \quad 0 < y < \frac{\pi}{4} \]
find \( \frac{dy}{dx} \) as a function of \( x \) in its simplest form.
\[(4)\]  
[2017 June Q7]

31. Given that
\[ y = \frac{\cos 2\theta}{1 + \sin 2\theta}, \quad -\frac{\pi}{4} < \theta < \frac{3\pi}{4} \]
show that
\[ \frac{dy}{d\theta} = \frac{\alpha}{1 + \sin 2\theta}, \quad -\frac{\pi}{4} < \theta < \frac{3\pi}{4} \]
where \( \alpha \) is a constant to be determined.
\[(4)\]  
[2014 June, IAL Q3]

32. (a) Use the identity for \( \sin(A + B) \) to prove that
\[ \sin 2A \equiv 2 \sin A \cos A \]
\[(2)\]

(b) Show that
\[ \frac{d}{dx} \left[ \ln \left( \tan \left( \frac{1}{2}x \right) \right) \right] = \csc x \]
\[(4)\]

A curve \( C \) has the equation
\[ y = \ln \left( \tan \left( \frac{1}{2}x \right) \right) - 3 \sin x, \quad 0 < x < \pi \]

(c) Find the \( x \) coordinates of the points on \( C \) where \( \frac{dy}{dx} = 0 \).
Give your answers to 3 decimal places.
\[(6)\]  
[2014 June, IAL Q10]
Figure 2 shows a sketch of part of the curve $C$ with equation

$$y = e^{a-3x} - 3e^{-x}, \quad x \in \mathbb{R}$$

where $a$ is a constant and $a > \ln 4$.

The curve $C$ has a turning point $P$ and crosses the $x$-axis at the point $Q$ as shown in Figure 2.

(a) Find, in terms of $a$, the coordinates of the point $P$.

(b) Find, in terms of $a$, the $x$ coordinate of the point $Q$.

(c) Sketch the curve with equation

$$y = \left|e^{a-3x} - 3e^{-x}\right|, \quad x \in \mathbb{R}, a > \ln 4$$

Show on your sketch the exact coordinates, in terms of $a$, of the points at which the curve meets or cuts the coordinate axes.

34. The curve $C$ has equation

$$y = \frac{3x-2}{(x-2)^2}, \quad x \neq 2$$

The point $P$ on $C$ has $x$ coordinate 3.

Find an equation of the normal to $C$ at the point $P$ in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are integers.
35.

Figure 1 shows a sketch of part of the curve with equation \( y = f(x) \), where

\[
f(x) = (2x - 5)e^x, \quad x \in \mathbb{R}
\]

The curve has a minimum turning point at \( A \).

(a) Use calculus to find the exact coordinates of \( A \).\( \quad (5) \)

Given that the equation \( f(x) = k \), where \( k \) is a constant, has exactly two roots,

(b) state the range of possible values of \( k \).\( \quad (2) \)

(c) Sketch the curve with equation \( y = |f(x)| \).

Indicate clearly on your sketch the coordinates of the points at which the curve crosses or meets the axes.\( \quad (3) \)

[2015 June, IAL Q3]
36. \[ g(x) = \frac{x^4 + x^3 - 7x^2 + 8x - 48}{x^2 + x - 12}, \quad x > 3, \quad x \in \mathbb{R} \]

(a) Given that

\[ \frac{x^4 + x^3 - 7x^2 + 8x - 48}{x^2 + x - 12} = x^2 + A + \frac{B}{x-3} \]

find the values of the constants \( A \) and \( B \). (4)

(b) Hence, or otherwise, find the equation of the tangent to the curve with equation \( y = g(x) \) at the point where \( x = 4 \). Give your answer in the form \( y = mx + c \), where \( m \) and \( c \) are constants to be determined. (5)

[2016 June, IAL Q4]

37. (i) Differentiate \( y = 5x^2 \ln 3x \), \( x > 0 \) (2)

(ii) Given that

\[ y = \frac{x}{\sin x + \cos x}, \quad \frac{3}{4} < x < \frac{3}{4} \]

show that

\[ \frac{dy}{dx} = \frac{(1+x)\sin x + (1-x)\cos x}{1+\sin 2x}, \quad \frac{3}{4} < x < \frac{3}{4} \] (4)

[2017 Jan, IAL Q6]