OCR Core Maths 3

Past paper questions
Trigonometry

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Trigonometry

- By definition $\sec\theta \equiv \dfrac{1}{\cos\theta}$, $\quad \operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$, $\quad \cot\theta \equiv \dfrac{1}{\tan\theta}$.
- If you get an equation where one of the new trig functions equals a constant, then just take the reciprocal of each side and solve *à la* C2. For example

$$\sec\theta = 5 \qquad \Rightarrow \qquad \frac{1}{\cos\theta} = 5 \qquad \Rightarrow \qquad \cos\theta = \frac{1}{5}.$$

- Know the graphs of $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$. Page 91/2 of your textbook.
- By dividing $\sin^2 x + \cos^2 x \equiv 1$ by $\sin^2 x$ and $\cos^2 x$ we can derive

$$1 + \cot^2 x \equiv \operatorname{cosec}^2 x \quad \text{and} \quad \tan^2 x + 1 \equiv \sec^2 x \quad \text{respectively.}$$

  These create a whole new family of equations that reduce to a quadratic in disguise. For example solve $3\cot^2\theta + 5\operatorname{cosec}\theta + 1 = 0$ in the range $0 \leqslant \theta \leqslant 2\pi$. Firstly note we will need to replace the $\cot^2\theta$ by $\operatorname{cosec}^2\theta - 1$ to reduce the equation to one trig function only.

$$\begin{aligned} 3\cot^2\theta + 5\operatorname{cosec}\theta + 1 &= 0 \\ 3(\operatorname{cosec}^2\theta - 1) + 5\operatorname{cosec}\theta + 1 &= 0 \\ 3\operatorname{cosec}^2\theta + 5\operatorname{cosec}\theta - 2 &= 0 \\ (3\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 2) &= 0 \\ \operatorname{cosec}\theta = \tfrac{1}{3} \quad \text{or} \quad \operatorname{cosec}\theta &= -2. \end{aligned}$$

  Therefore $\sin\theta = 3$ which has no solutions, or $\sin\theta = -\frac{1}{2}$ which gives the solutions $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

- You must know, and be able to apply, the compound angle formulae:

$$\begin{aligned} \sin(A \pm B) &= \sin A\cos B \pm \cos A\sin B, \\ \cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B, \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}. \end{aligned}$$

- You must know, and be able to apply, the double angle formulae (derived by setting $A = B$ in the compound angle formulae above):

$$\begin{aligned}\sin 2A &= 2\sin A\cos A,\\ \cos 2A &= \cos^2 A - \sin^2 A,\\ &= 2\cos^2 A - 1,\\ &= 1 - 2\sin^2 A,\\ \tan 2A &= \frac{2\tan A}{1-\tan^2 A}.\end{aligned}$$

  Notice there are three versions of the double angle formula for $\cos 2A$; you need to *think hard* about which form you will need for the question you are solving. You will hardly ever need the fist of the three ($\cos^2\theta - \sin^2\theta$) because it involves two different trig functions; the aim is, usually, to get only one.

- You must be able to convert form the form $a\cos\theta \pm b\sin\theta$ into either $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$; the question will specify which. This then enables us to solve equations of the form

$$a\cos\theta \pm b\sin\theta = \text{constant}.$$

  For example express $3\cos\theta - 5\sin\theta$ in the form $R\cos(\theta+\alpha)$. Always start by looking at the coefficients of $\cos\theta$ and $\sin\theta$ in the original expression; here they are 3 and 5 (ignore the sign). Sum their squares and square root (like Pythagoras) and factorise out:

$$3\cos\theta - 5\sin\theta \equiv \sqrt{34}\left[\frac{3}{\sqrt{34}}\cos\theta - \frac{5}{\sqrt{34}}\sin\theta\right].$$

  Next consider the form of the answer we are aiming for; here "$R\cos(\theta+\alpha)$". The expansion of "$R\cos(\theta+\alpha)$" is "$R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$". Comparing

$$\sqrt{34}\left[\frac{3}{\sqrt{34}}\cos\theta - \frac{5}{\sqrt{34}}\sin\theta\right] \quad \text{with} \quad R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

  we see instantly $R = \sqrt{34}$. We also require $\frac{3}{\sqrt{34}} = \cos\alpha$ and $\frac{5}{\sqrt{34}} = \sin\alpha$; solving either of those two we find $\alpha = 59.0°$ (to 1 d.p). Therefore

$$3\cos\theta - 5\sin\theta \equiv \sqrt{34}\cos(\theta + 59.0°).$$

- The trig functions all have inverses if we restrict the domain. The conventional restrictions to allow inversion are

FUNCTION	DOMAIN	DOMAIN
$y = \sin x$	$-90° \leqslant x \leqslant 90°$	$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$
$y = \cos x$	$0° \leqslant x \leqslant 180°$	$0 \leqslant x \leqslant \pi$
$y = \tan x$	$-90° < x < 90°$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

  Know what the graphs of $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$ look like.

1.

(i) Express $3\sin\theta + 2\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. [3]

(ii) Hence solve the equation $3\sin\theta + 2\cos\theta = \frac{7}{2}$, giving all solutions for which $0^\circ < \theta < 360^\circ$. [5]

Q5 June 2005

2.

(i) Write down the formula for $\cos 2x$ in terms of $\cos x$. [1]

(ii) Prove the identity $\dfrac{4\cos 2x}{1 + \cos 2x} \equiv 4 - 2\sec^2 x$. [3]

(iii) Solve, for $0 < x < 2\pi$, the equation $\dfrac{4\cos 2x}{1 + \cos 2x} = 3\tan x - 7$. [5]

Q7 June 2005

3.

Solve, for $0^\circ < \theta < 360^\circ$, the equation $\sec^2\theta = 4\tan\theta - 2$. [5]

Q2 Jan 2006

4.

(i) By first writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$ [4]

(ii) Determine the greatest possible value of

$$9\sin\left(\tfrac{10}{3}\alpha\right) - 12\sin^3\left(\tfrac{10}{3}\alpha\right),$$

and find the smallest positive value of α (in degrees) for which that greatest value occurs. [3]

(iii) Solve, for $0^\circ < \beta < 90^\circ$, the equation $3\sin 6\beta \operatorname{cosec} 2\beta = 4$. [6]

Q9 Jan 2006

5.

(i) Write down the identity expressing $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$. [1]

(ii) Given that $\sin\alpha = \frac{1}{4}$ and α is acute, show that $\sin 2\alpha = \frac{1}{8}\sqrt{15}$. [3]

(iii) Solve, for $0° < \beta < 90°$, the equation $5\sin 2\beta \sec\beta = 3$. [3]

Q5 June 2006

6.

(i) Express $5\cos x + 12\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

(ii) Hence give details of a pair of transformations which transforms the curve $y = \cos x$ to the curve $y = 5\cos x + 12\sin x$. [3]

(iii) Solve, for $0° < x < 360°$, the equation $5\cos x + 12\sin x = 2$, giving your answers correct to the nearest $0.1°$. [5]

Q6 June 2006

7.

It is given that θ is the acute angle such that $\sin\theta = \frac{12}{13}$. Find the exact value of

(i) $\cot\theta$, [2]

(ii) $\cos 2\theta$. [3]

Q2 Jan 2007

8.

(i) Express $4\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

(ii) Hence solve the equation $4\cos\theta - \sin\theta = 2$, giving all solutions for which $-180° < \theta < 180°$. [5]

Q5 Jan 2007

9.

(i) Sketch the graph of $y = \sec x$ for $0 \leqslant x \leqslant 2\pi$. [2]

(ii) Solve the equation $\sec x = 3$ for $0 \leqslant x \leqslant 2\pi$, giving the roots correct to 3 significant figures. [3]

(iii) Solve the equation $\sec\theta = 5\operatorname{cosec}\theta$ for $0 \leqslant \theta \leqslant 2\pi$, giving the roots correct to 3 significant figures. [4]

Q7 June 2007

10.

(a) Solve, for $0^\circ < \alpha < 180^\circ$, the equation $\sec \frac{1}{2}\alpha = 4$. [3]

(b) Solve, for $0^\circ < \beta < 180^\circ$, the equation $\tan \beta = 7 \cot \beta$. [4]

Q3 Jan 2008

11.

(i) Prove the identity

$$\tan(\theta + 60^\circ)\tan(\theta - 60^\circ) \equiv \frac{\tan^2\theta - 3}{1 - 3\tan^2\theta}.$$ [4]

(ii) Solve, for $0^\circ < \theta < 180^\circ$, the equation

$$\tan(\theta + 60^\circ)\tan(\theta - 60^\circ) = 4\sec^2\theta - 3,$$

giving your answers correct to the nearest 0.1°. [5]

(iii) Show that, for all values of the constant k, the equation

$$\tan(\theta + 60^\circ)\tan(\theta - 60^\circ) = k^2$$

has two roots in the interval $0^\circ < \theta < 180^\circ$. [3]

Q9 June 2007

12.

(i) Use the identity for $\cos(A + B)$ to prove that

$$4\cos(\theta + 60^\circ)\cos(\theta + 30^\circ) \equiv \sqrt{3} - 2\sin 2\theta.$$ [4]

(ii) Hence find the exact value of $4\cos 82.5^\circ \cos 52.5^\circ$. [2]

(iii) Solve, for $0^\circ < \theta < 90^\circ$, the equation $4\cos(\theta + 60^\circ)\cos(\theta + 30^\circ) = 1$. [3]

(iv) Given that there are no values of θ which satisfy the equation

$$4\cos(\theta + 60^\circ)\cos(\theta + 30^\circ) = k,$$

determine the set of values of the constant k. [3]

Q9 Jan 2008

13.

(a) Express $\tan 2\alpha$ in terms of $\tan \alpha$ and hence solve, for $0^\circ < \alpha < 180^\circ$, the equation

$$\tan 2\alpha \tan \alpha = 8.$$ [6]

(b) Given that β is the acute angle such that $\sin \beta = \frac{6}{7}$, find the exact value of

(i) $\operatorname{cosec} \beta$, [1]

(ii) $\cot^2 \beta$. [2]

Q5 June 2008

14.

The expression $T(\theta)$ is defined for θ in degrees by

$$T(\theta) = 3\cos(\theta - 60^\circ) + 2\cos(\theta + 60^\circ).$$

(i) Express $T(\theta)$ in the form $A \sin \theta + B \cos \theta$, giving the exact values of the constants A and B. [3]

(ii) Hence express $T(\theta)$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. [3]

(iii) Find the smallest positive value of θ such that $T(\theta) + 1 = 0$. [4]

Q8 June 2008

15.

(i) Express $2\tan^2 \theta - \dfrac{1}{\cos \theta}$ in terms of $\sec \theta$. **[3]**

(ii) Hence solve, for $0^\circ < \theta < 360^\circ$, the equation

$$2\tan^2 \theta - \frac{1}{\cos \theta} = 4.$$ **[4]**

Q3 Jan 2009

16.

(i) By first expanding $\cos(2\theta + \theta)$, prove that

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta. \qquad \textbf{[4]}$$

(ii) Hence prove that

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1. \qquad \textbf{[3]}$$

(iii) Show that the only solutions of the equation

$$1 + \cos 6\theta = 18\cos^2\theta$$

are odd multiples of 90°. **[5]**

Q9 Jan 2009

17.

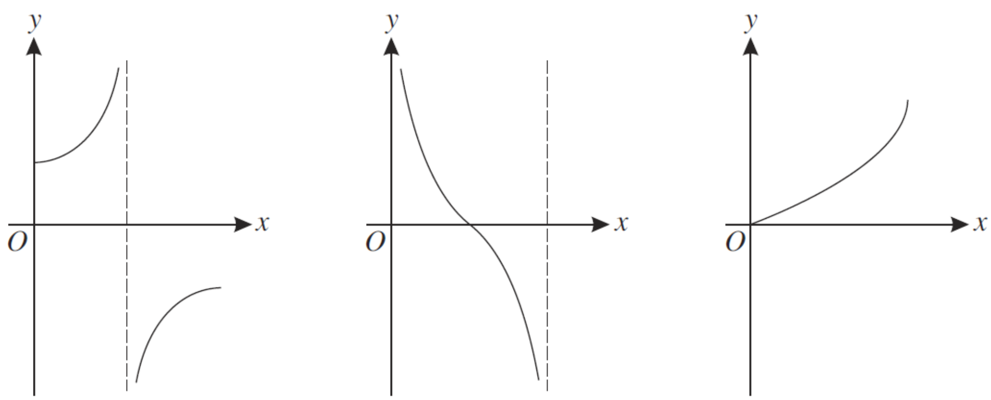

Fig. 1 **Fig. 2** **Fig. 3**

Each diagram above shows part of a curve, the equation of which is one of the following:

$$y = \sin^{-1} x, \quad y = \cos^{-1} x, \quad y = \tan^{-1} x, \quad y = \sec x, \quad y = \operatorname{cosec} x, \quad y = \cot x.$$

State which equation corresponds to

(i) Fig. 1, **[1]**

(ii) Fig. 2, **[1]**

(iii) Fig. 3. **[1]**

Q1 June 2009

18.

The angles α and β are such that

$$\tan\alpha = m + 2 \quad \text{and} \quad \tan\beta = m,$$

where m is a constant.

(i) Given that $\sec^2\alpha - \sec^2\beta = 16$, find the value of m. **[3]**

(ii) Hence find the exact value of $\tan(\alpha + \beta)$. **[3]**

Q3 June 2009

19.

(i) Express the equation $\operatorname{cosec}\theta(3\cos 2\theta + 7) + 11 = 0$ in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are constants. **[3]**

(ii) Hence solve, for $-180^\circ < \theta < 180^\circ$, the equation $\operatorname{cosec}\theta(3\cos 2\theta + 7) + 11 = 0$. **[3]**

Q3 June 2010

20.

(i) Express $8\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. **[3]**

(ii) Hence

(a) solve, for $0^\circ < \theta < 360^\circ$, the equation $8\sin\theta - 6\cos\theta = 9$, **[4]**

(b) find the greatest possible value of

$$32\sin x - 24\cos x - (16\sin y - 12\cos y)$$

as the angles x and y vary. **[3]**

Q7 June 2009

21.

The angle θ is such that $0^\circ < \theta < 90^\circ$.

(i) Given that θ satisfies the equation $6\sin 2\theta = 5\cos\theta$, find the exact value of $\sin\theta$. **[3]**

(ii) Given instead that θ satisfies the equation $8\cos\theta\operatorname{cosec}^2\theta = 3$, find the exact value of $\cos\theta$. **[5]**

Q2 Jan 2010

22.

The value of $\tan 10^\circ$ is denoted by p. Find, in terms of p, the value of

(i) $\tan 55^\circ$, **[3]**

(ii) $\tan 5^\circ$, **[4]**

(iii) $\tan\theta$, where θ satisfies the equation $3\sin(\theta + 10^\circ) = 7\cos(\theta - 10^\circ)$. **[5]**

Q9 Jan 2010

23.

(i) Express $3\cos x + 3\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. **[3]**

(ii) The expression $\mathrm{T}(x)$ is defined by $\mathrm{T}(x) = \dfrac{8}{3\cos x + 3\sin x}$.

(a) Determine a value of x for which $\mathrm{T}(x)$ is not defined. **[2]**

(b) Find the smallest positive value of x satisfying $\mathrm{T}(3x) = \frac{8}{9}\sqrt{6}$, giving your answer in an exact form. **[4]**

Q8 June 2010

24.

(i) Express $24\sin\theta + 7\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. **[3]**

(ii) Hence solve the equation $24\sin\theta + 7\cos\theta = 12$ for $0^\circ < \theta < 360^\circ$. **[4]**

Q4 Jan 2011

25.

(a) **(i)** Sketch the graph of $y = \operatorname{cosec} x$ for $0 < x < 4\pi$. **[3]**

(ii) It is given that $\operatorname{cosec}\alpha = \operatorname{cosec}\beta$, where $\frac{1}{2}\pi < \alpha < \pi$ and $2\pi < \beta < \frac{5}{2}\pi$. By using your sketch, or otherwise, express β in terms of α. **[2]**

(b) **(i)** Write down the identity giving $\tan 2\theta$ in terms of $\tan\theta$. **[1]**

(ii) Given that $\cot\phi = 4$, find the exact value of $\tan\phi\cot 2\phi\tan 4\phi$, showing all your working. **[6]**

Q8 Jan 2011

26.

(a) Given that $7\sin 2\alpha = 3\sin\alpha$, where $0^\circ < \alpha < 90^\circ$, find the exact value of $\cos\alpha$. **[3]**

(b) Given that $3\cos 2\beta + 19\cos\beta + 13 = 0$, where $90^\circ < \beta < 180^\circ$, find the exact value of $\sec\beta$. **[5]**

Q3 June 2011

27.

(i) Prove that $\dfrac{\sin(\theta-\alpha)+3\sin\theta+\sin(\theta+\alpha)}{\cos(\theta-\alpha)+3\cos\theta+\cos(\theta+\alpha)} \equiv \tan\theta$ for all values of α. **[5]**

(ii) Find the exact value of $\dfrac{4\sin 149^\circ + 12\sin 150^\circ + 4\sin 151^\circ}{3\cos 149^\circ + 9\cos 150^\circ + 3\cos 151^\circ}$. **[3]**

(iii) It is given that k is a positive constant. Solve, for $0^\circ < \theta < 60^\circ$ and in terms of k, the equation

$$\frac{\sin(6\theta-15^\circ)+3\sin 6\theta+\sin(6\theta+15^\circ)}{\cos(6\theta-15^\circ)+3\cos 6\theta+\cos(6\theta+15^\circ)} = k.$$ **[4]**

Q9 June 2011

28.

The acute angles α and β are such that

$$2\cot\alpha = 1 \quad \text{and} \quad 24 + \sec^2\beta = 10\tan\beta.$$

(i) State the value of $\tan\alpha$ and determine the value of $\tan\beta$. **[4]**

(ii) Hence find the exact value of $\tan(\alpha+\beta)$. **[3]**

Q4 Jan 2012

29.

(i) Express $\cos 4\theta$ in terms of $\sin 2\theta$ and hence show that $\cos 4\theta$ can be expressed in the form $1 - k\sin^2\theta\cos^2\theta$, where k is a constant to be determined. **[3]**

(ii) Hence find the exact value of $\sin^2(\frac{1}{24}\pi)\cos^2(\frac{1}{24}\pi)$. **[2]**

(iii) By expressing $2\cos^2 2\theta - \frac{8}{3}\sin^2\theta\cos^2\theta$ in terms of $\cos 4\theta$, find the greatest and least possible values of

$$2\cos^2 2\theta - \tfrac{8}{3}\sin^2\theta\cos^2\theta$$

as θ varies. **[5]**

Q8 Jan 2012

30.

It is given that θ is the acute angle such that $\sec\theta\sin\theta = 36\cot\theta$.

(i) Show that $\tan\theta = 6$. **[3]**

(ii) Hence, using an appropriate formula in each case, find the exact value of

(a) $\tan(\theta - 45°)$, **[2]**

(b) $\tan 2\theta$. **[2]**

Q3 June 2012

31.

(i) Express $3\sin\theta + 4\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. **[3]**

(ii) Hence

(a) solve the equation $3\sin\theta + 4\cos\theta + 1 = 0$, giving all solutions for which $-180° < \theta < 180°$, **[4]**

(b) find the values of the positive constants k and c such that

$$-37 \leqslant k(3\sin\theta + 4\cos\theta) + c \leqslant 43$$

for all values of θ. **[4]**

Q8 June 2012

32.

The acute angle A is such that $\tan A = 2$.

(i) Find the exact value of $\operatorname{cosec} A$. **[2]**

(ii) The angle B is such that $\tan(A + B) = 3$. Using an appropriate identity, find the exact value of $\tan B$. **[3]**

Q2 Jan 2013

33.

(i) Prove that

$$\cos^2(\theta + 45°) - \tfrac{1}{2}(\cos 2\theta - \sin 2\theta) \equiv \sin^2\theta. \quad \textbf{[4]}$$

(ii) Hence solve the equation

$$6\cos^2(\tfrac{1}{2}\theta + 45°) - 3(\cos\theta - \sin\theta) = 2$$

for $-90° < \theta < 90°$. **[3]**

(iii) It is given that there are two values of θ, where $-90° < \theta < 90°$, satisfying the equation

$$6\cos^2(\tfrac{1}{3}\theta + 45°) - 3(\cos\tfrac{2}{3}\theta - \sin\tfrac{2}{3}\theta) = k,$$

where k is a constant. Find the set of possible values of k. **[3]**

Q9 Jan 2013

34.

Using an appropriate identity in each case, find the possible values of

(i) $\sin\alpha$ given that $4\cos 2\alpha = \sin^2\alpha$, **[3]**

(ii) $\sec\beta$ given that $2\tan^2\beta = 3 + 9\sec\beta$. **[4]**

Q2 June 2013

35.

(i) Express $4\cos\theta - 2\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. **[3]**

(ii) Hence

(a) solve the equation $4\cos\theta - 2\sin\theta = 3$ for $0° < \theta < 360°$, **[4]**

(b) determine the greatest and least values of

$$25 - (4\cos\theta - 2\sin\theta)^2$$

as θ varies, and, in each case, find the smallest positive value of θ for which that value occurs. **[5]**

Q8 June 2013

36.

By first using appropriate identities, solve the equation

$$5\cos 2\theta \operatorname{cosec}\theta = 2$$

for $0° < \theta < 180°$. **[6]**

Q2 June 2014

37.

(i) Express $5\cos(\theta - 60°) + 3\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. **[4]**

(ii) Hence

(a) give details of the transformations needed to transform the curve $y = 5\cos(\theta - 60°) + 3\cos\theta$ to the curve $y = \sin\theta$, **[3]**

(b) find the smallest positive value of β satisfying the equation

$$5\cos\left(\tfrac{1}{3}\beta - 40°\right) + 3\cos\left(\tfrac{1}{3}\beta + 20°\right) = 3.$$ **[5]**

Q9 June 2014

38.

It is given that θ is the acute angle such that $\cot\theta = 4$. Without using a calculator, find the exact value of

(i) $\tan(\theta + 45°)$, **[3]**

(ii) $\operatorname{cosec}\theta$. **[2]**

Q2 June 2015

39.

It is given that $f(\theta) = \sin(\theta + 30°) + \cos(\theta + 60°)$.

(i) Show that $f(\theta) = \cos\theta$. Hence show that

$$f(4\theta) + 4f(2\theta) \equiv 8\cos^4\theta - 3.$$ **[6]**

(ii) Hence

(a) determine the greatest and least values of $\dfrac{1}{f(4\theta) + 4f(2\theta) + 7}$ as θ varies, **[3]**

(b) solve the equation

$$\sin(12\alpha + 30°) + \cos(12\alpha + 60°) + 4\sin(6\alpha + 30°) + 4\cos(6\alpha + 60°) = 1$$

for $0° < \alpha < 60°$. **[4]**

Q9 June 2015