OCR Further Pure 1

Past paper questions
Series

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Summing Series

- You must know the important results:

$$\sum_{r=1}^{n} 1 = 1 + 1 + 1 + \cdots + 1 = n,$$
$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1),$$
$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1),$$
$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

- Also know the important properties (for constant λ and μ)

$$\sum_{r=1}^{n}(\lambda f(r) \pm \mu g(r)) = \lambda \sum_{r=1}^{n} f(r) \pm \mu \sum_{r=1}^{n} g(r).$$

However beware of these!

$$\sum_{r=1}^{n}(f(r) \times g(r)) \neq \sum_{r=1}^{n} f(r) \times \sum_{r=1}^{n} g(r) \qquad \text{and} \qquad \sum_{r=1}^{n}\left(\frac{f(r)}{g(r)}\right) \neq \frac{\sum_{r=1}^{n} f(r)}{\sum_{r=1}^{n} g(r)}.$$

To apply the first of these is equivalent to the heinous crime of $(a+b+c)^2 = a^2+b^2+c^2$!!!

- These must be applied in cases such as:

$$\begin{aligned}\sum_{r=1}^{n}(4r^2 - 2r + 3) &= 4\sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r + 3\sum_{r=1}^{n} 1 \\ &= \tfrac{2}{3}n(n+1)(2n+1) - n(n+1) + 3n \\ &= \tfrac{n}{3}[2(n+1)(2n+1) - 3(n+1) + 9] \\ &= \tfrac{n}{3}(4n^2 + 3n + 8).\end{aligned}$$

- The *method of differences* can be used to sum certain expressions where cancellation occurs when the sum is written out. For example find $\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$. Write the sum out, starting a new line for each value of r and you should see that some nice cancelling occurs;

$$\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right) = \frac{1}{3}-\frac{1}{5}$$
$$+\frac{1}{5}-\frac{1}{7}$$
$$+\frac{1}{7}-\frac{1}{9}$$
$$\vdots$$
$$+\frac{1}{2n+1}-\frac{1}{2n+3}.$$

  You can see that everything cancels except the $\frac{1}{3}$ and the $\frac{1}{2n+3}$ so

$$\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right) = \frac{1}{3}-\frac{1}{2n+3}.$$

  It is usually best *not* to combine these terms together into one fraction in order to make it easier to see if there is a sum to infinity.

- A sum to infinity exists if the expression for the sum to n has a finite limit as $n \to \infty$. In the above example it does, so

$$\sum_{r=1}^{\infty}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right) = \lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2n+3}\right) = \frac{1}{3}.$$

- Questions of this sort invariably start "Show that $f(r) - g(r) = h(r)$", and then ask you to sum $h(r)$; this, clearly, is the same as summing $f(r) - g(r) \Rightarrow$ 'method of differences'.

  - If the sum starts from a number other than 1 then you can use the trick (which should be obvious)

$$\sum_{r=a}^{n}(\text{something}) = \sum_{r=1}^{n}(\text{something}) - \sum_{r=1}^{a-1}(\text{something}).$$

1.

Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n}(6r^2 + 2r + 1) = n(2n^2 + 4n + 3). \qquad [6]$$

(June 2005, Q1)

2.

(i) Show that

$$\frac{r+1}{r+2} - \frac{r}{r+1} = \frac{1}{(r+1)(r+2)}. \qquad [2]$$

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots + \frac{1}{(n+1)(n+2)}. \qquad [4]$$

(iii) Hence write down the value of $\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+2)}$. [1]

(June 2005, Q5)

3.

Use the standard results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ to show that, for all positive integers n,

$$\sum_{r=1}^{n}(8r^3 - 6r^2 + 2r) = 2n^3(n+1). \qquad [6]$$

(Jan 2006, Q5)

4.

(i) Show that $\frac{1}{r} - \frac{1}{r+2} = \frac{2}{r(r+2)}$. [2]

(ii) Hence find an expression, in terms of n, for

$$\frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \dots + \frac{2}{n(n+2)}. \qquad [5]$$

(iii) Hence find the value of

(a) $\sum_{r=1}^{\infty} \frac{2}{r(r+2)}$, [1]

(b) $\sum_{r=n+1}^{\infty} \frac{2}{r(r+2)}$. [2]

(Jan 2006, Q9)

5.

Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} (r^3 + r^2) = \tfrac{1}{12}n(n+1)(n+2)(3n+1). \qquad [5]$$

(June 2006, Q5)

6.

(i) Use the method of differences to show that

$$\sum_{r=1}^{n} \left\{(r+1)^3 - r^3\right\} = (n+1)^3 - 1. \qquad [2]$$

(ii) Show that $(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1$. [2]

(iii) Use the results in parts **(i)** and **(ii)** and the standard result for $\sum_{r=1}^{n} r$ to show that

$$3\sum_{r=1}^{n} r^2 = \tfrac{1}{2}n(n+1)(2n+1). \qquad [6]$$

(June 2006, Q9)

7.

Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to find

$$\sum_{r=1}^{n} r(r-1)(r+1),$$

expressing your answer in a fully factorised form. [6]

(Jan 2007, Q3)

8.

(i) Show that $(r+2)! - (r+1)! = (r+1)^2 \times r!$. [3]

(ii) Hence find an expression, in terms of n, for

$$2^2 \times 1! \; + \; 3^2 \times 2! \; + \; 4^2 \times 3! \; + \; \ldots \; + \; (n+1)^2 \times n!. \qquad [4]$$

(iii) State, giving a brief reason, whether the series

$$2^2 \times 1! \; + \; 3^2 \times 2! \; + \; 4^2 \times 3! \; + \; \ldots$$

converges. [1]

(Jan 2007, Q8)

9.

Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n}(3r^2 - 3r + 1) = n^3.$$ [6]

(June 2007, Q3)

10.

(i) Show that

$$\frac{1}{r} - \frac{1}{r+1} = \frac{1}{r(r+1)}.$$ [1]

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{n(n+1)}.$$ [3]

(iii) Hence find the value of $\sum_{r=n+1}^{\infty} \frac{1}{r(r+1)}$. [3]

(June 2007, Q5)

11.

Given that $\sum_{r=1}^{n}(ar^2 + b) \equiv n(2n^2 + 3n - 2)$, find the values of the constants a and b. [5]

(Jan 2008, Q2)

12.

(i) Show that $\frac{2}{r} - \frac{1}{r+1} - \frac{1}{r+2} = \frac{3r+4}{r(r+1)(r+2)}$. [2]

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{3r+4}{r(r+1)(r+2)}.$$ [6]

(iii) Hence write down the value of $\sum_{r=1}^{\infty} \frac{3r+4}{r(r+1)(r+2)}$. [1]

(iv) Given that $\sum_{r=N+1}^{\infty} \frac{3r+4}{r(r+1)(r+2)} = \frac{7}{10}$, find the value of N. [4]

(Jan 2008, Q10)

13.

(i) Show that $\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}$. [2]

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!}.$$ [4]

(June 2008, Q3)

14.

Find $\sum_{r=1}^{n} r^2(r-1)$, expressing your answer in a fully factorised form. [6]

(June 2008, Q5)

15.

Find $\sum_{r=1}^{n} (4r^3 + 6r^2 + 2r)$, expressing your answer in a fully factorised form. **[6]**

(Jan 2009, Q3)

16.

(i) Show that $\frac{1}{2r-3} - \frac{1}{2r+1} = \frac{4}{4r^2 - 4r - 3}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=2}^{n} \frac{4}{4r^2 - 4r - 3}.$$ **[6]**

(iii) Show that $\sum_{r=2}^{\infty} \frac{4}{4r^2 - 4r - 3} = \frac{4}{3}$. **[1]**

(Jan 2009, Q9)

17.

Evaluate $\sum_{r=101}^{250} r^3$. **[3]**

(June 2009, Q1)

18.

(i) Use the method of differences to show that

$$\sum_{r=1}^{n}\{(r+1)^4 - r^4\} = (n+1)^4 - 1. \qquad \textbf{[2]}$$

(ii) Show that $(r+1)^4 - r^4 \equiv 4r^3 + 6r^2 + 4r + 1$. **[2]**

(iii) Hence show that

$$4\sum_{r=1}^{n} r^3 = n^2(n+1)^2. \qquad \textbf{[6]}$$

(June 2009, Q7)

19.

Find $\sum_{r=1}^{n} r(r+1)(r-2)$, expressing your answer in a fully factorised form. **[6]**

(Jan 2010, Q4)

20.

(i) Show that $\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}$. **[1]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$. **[4]**

(iii) Find $\sum_{r=2}^{\infty} \frac{2r+1}{r^2(r+1)^2}$. **[2]**

(Jan 2010, Q7)

21.

Find $\sum_{r=1}^{n}(2r-1)^2$, expressing your answer in a fully factorised form. **[6]**

(June 2010, Q3)

22.

(i) Show that $\frac{1}{\sqrt{r+2}+\sqrt{r}} \equiv \frac{\sqrt{r+2}-\sqrt{r}}{2}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r+2}+\sqrt{r}}. \qquad \textbf{[6]}$$

(iii) State, giving a brief reason, whether the series $\sum_{r=1}^{\infty} \frac{1}{\sqrt{r+2}+\sqrt{r}}$ converges. **[1]**

(June 2010, Q8)

23.

Given that $\sum_{r=1}^{n}(ar^3 + br) \equiv n(n-1)(n+1)(n+2)$, find the values of the constants a and b. **[6]**

(Jan 2011, Q4)

24.

(i) Show that $\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2} \equiv \frac{2}{r(r+1)(r+2)}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)}.$$ **[6]**

(iii) Show that $\sum_{r=n+1}^{\infty} \frac{2}{r(r+1)(r+2)} = \frac{1}{(n+1)(n+2)}$. **[3]**

(Jan 2011, Q10)

25.

Find $\sum_{r=1}^{2n}(3r^2 - \frac{1}{2})$, expressing your answer in a fully factorised form. **[6]**

(June 2011, Q7)

26.

(i) Show that $\frac{1}{r-1} - \frac{1}{r+1} \equiv \frac{2}{r^2-1}$. **[1]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=2}^{n} \frac{2}{r^2-1}$. **[5]**

(iii) Find the value of $\sum_{r=1000}^{\infty} \frac{2}{r^2-1}$. **[3]**

(June 2011, Q7)

27.

Find $\sum_{r=1}^{n} r(r^2 - 3)$, expressing your answer in a fully factorised form. **[6]**

(Jan 2012, Q4)

28.

(i) Show that $\frac{r}{r+1} - \frac{r-1}{r} \equiv \frac{1}{r(r+1)}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)}.$$ **[4]**

(iii) Hence find $\sum_{r=n+1}^{\infty} \frac{1}{r(r+1)}$. **[2]**

(Jan 2012, Q8)

29.

Find $\sum_{r=1}^{n} (3r^2 - 3r + 2)$, expressing your answer in a fully factorised form. **[7]**

(June 2012, Q4)

30.

(i) Show that $\frac{1}{r} - \frac{1}{r+2} \equiv \frac{2}{r(r+2)}$. **[1]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=1}^{n} \frac{2}{r(r+2)}$. **[6]**

(iii) Given that $\sum_{r=N+1}^{\infty} \frac{2}{r(r+2)} = \frac{11}{30}$, find the value of N. **[4]**

(June 2012, Q8)

31.

Find $\sum_{r=1}^{n} (r-1)(r+1)$, giving your answer in a fully factorised form. **[6]**

(Jan 2013, Q2)

32.

(i) Show that $\frac{1}{r} - \frac{3}{r+1} + \frac{2}{r+2} \equiv \frac{2-r}{r(r+1)(r+2)}$. **[2]**

(ii) Hence show that $\sum_{r=1}^{n} \frac{2-r}{r(r+1)(r+2)} = \frac{n}{(n+1)(n+2)}$. **[5]**

(iii) Find the value of $\sum_{r=2}^{\infty} \frac{2-r}{r(r+1)(r+2)}$. **[2]**

(Jan 2013, Q8)

33.

Find $\sum_{r=1}^{n} (4r^3 - 3r^2 + r)$, giving your answer in a fully factorised form. **[6]**

(June 2013, Q5)

34.

(i) Show that $\frac{1}{3r-1} - \frac{1}{3r+2} \equiv \frac{3}{(3r-1)(3r+2)}$. **[2]**

(ii) Hence show that $\sum_{r=1}^{2n} \frac{1}{(3r-1)(3r+2)} = \frac{n}{2(3n+1)}$. **[6]**

(June 2013, Q9)

35.

(i) Show that $\frac{1}{r^2} - \frac{1}{(r+2)^2} \equiv \frac{4(r+1)}{r^2(r+2)^2}$. **[2]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=1}^{n} \frac{4(r+1)}{r^2(r+2)^2}$. **[6]**

(iii) Find $\sum_{r=5}^{\infty} \frac{4(r+1)}{r^2(r+2)^2}$, giving your answer in the form $\frac{p}{q}$ where p and q are integers. **[2]**

(June 2014, Q6)

36.

(i) Show that $\sum_{r=n}^{2n} r^3 = \frac{3}{4}n^2(n+1)(5n+1)$. **[4]**

(ii) Hence find $\sum_{r=n}^{2n} r(r^2-2)$, giving your answer in a fully factorised form. **[5]**

(June 2014, Q8)

37.

Find $\sum_{r=1}^{n} (3r^2-5)$, expressing your answer in a fully factorised form. **[4]**

(June 2015, Q2)

38.

(i) Show that $\frac{3}{r-1} - \frac{2}{r} - \frac{1}{r+1} \equiv \frac{4r+2}{r(r^2-1)}$. **[2]**

(ii) Hence find an expression, in terms of n, for $\sum_{r=2}^{n} \frac{4r+2}{r(r^2-1)}$. **[6]**

(iii) Hence find the value of $\sum_{r=4}^{\infty} \frac{4r+2}{r(r^2-1)}$. **[2]**

(June 2015, Q8)