OCR

Further Maths

Pure Core Mathematics

Year 1

Further Algebra.

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Roots Of Equations

- By considering the general quadratic equation $ax^2+bx+c=0$ we re-write it as $x^2+\frac{b}{a}x+\frac{c}{a}=0$. Quadratics can be factorised into two linear factors $(x-\alpha)(x-\beta)$. By equating the two we find

$$(x-\alpha)(x-\beta) = x^2+\frac{b}{a}x+\frac{c}{a}$$
$$x^2-(\alpha+\beta)x+\alpha\beta = x^2+\frac{b}{a}x+\frac{c}{a}.$$

So we see that the sum of the roots of a quadratic is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$.

- By two tedious derivations (that you *should* do for yourself) similar to the one above we find that for the cubic $(ax^3+bx^2+cx+d=0)$ and the quartic $(ax^4+bx^3+cx^2+dx+e=0)$ the following:

Quadratics	Cubics	Quartics
$\alpha+\beta=-\frac{b}{a}$	$\alpha+\beta+\gamma=-\frac{b}{a}$	$\alpha+\beta+\gamma+\delta=-\frac{b}{a}$
$\alpha\beta=\frac{c}{a},$	$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$	$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}$
	$\alpha\beta\gamma=-\frac{d}{a},$	$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}$
		$\alpha\beta\gamma\delta=\frac{e}{a}.$

For your exam you only need quadratics and cubics, but the pattern continues fairly easily to quartics, quintics and beyond.

- For speed of writing we use the following shorthand:

$$\alpha+\beta+\gamma \equiv \sum\alpha \qquad \text{and} \qquad \alpha\beta+\alpha\gamma+\beta\gamma \equiv \sum\alpha\beta.$$

- We can therefore find properties of roots from equations without having to solve the equations themselves.

  For example from $2x^2-3x-6=0$ I can say that $\alpha\beta=\frac{-6}{2}=-3$ and $\alpha+\beta=\frac{3}{2}$.

  For example from $2x^3-4x^2-3x+6=0$ I can say that $\alpha\beta\gamma=\frac{-6}{2}=-3$, $\sum\alpha\beta=-\frac{3}{2}$ and $\sum\alpha=\frac{4}{2}=2$. Watch those signs!

1.

(a) The quadratic equation $x^2 - 2x + 4 = 0$ has roots α and β.

(i) Write down the values of $\alpha + \beta$ and $\alpha\beta$. [2]

(ii) Show that $\alpha^2 + \beta^2 = -4$. [2]

(iii) Hence find a quadratic equation which has roots α^2 and β^2. [3]

(b) The cubic equation $x^3 - 12x^2 + ax - 48 = 0$ has roots p, $2p$ and $3p$.

(i) Find the value of p. [2]

(ii) Hence find the value of a. [2]

(Q8, June 2005)

2.

Use the substitution $x = u + 2$ to find the exact value of the real root of the equation

$$x^3 - 6x^2 + 12x - 13 = 0.$$ [5]

(Q4, Jan 2006)

3.

The roots of the equation

$$x^3 - 9x^2 + 27x - 29 = 0$$

are denoted by α, β and γ, where α is real and β and γ are complex.

(i) Write down the value of $\alpha + \beta + \gamma$. [1]

(ii) It is given that $\beta = p + iq$, where $q > 0$. Find the value of p, in terms of α. [4]

(iii) Write down the value of $\alpha\beta\gamma$. [1]

(iv) Find the value of q, in terms of α only. [5]

(Q10, Jan 2006)

4.

The cubic equation $x^3 - 2x^2 + 3x + 4 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. [3]

The cubic equation $x^3 + px^2 + 10x + q = 0$, where p and q are constants, has roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$.

(ii) Find the value of p. [3]

(iii) Find the value of q. [5]

(Q10, June 2006)

5.

The quadratic equation $x^2 + 5x + 10 = 0$ has roots α and β.

(i) Write down the values of $\alpha + \beta$ and $\alpha\beta$. [2]

(ii) Show that $\alpha^2 + \beta^2 = 5$. [2]

(iii) Hence find a quadratic equation which has roots $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$. [4]

(Q7, Jan 2007)

6.

The cubic equation $3x^3 - 9x^2 + 6x + 2 = 0$ has roots α, β and γ.

(i) (a) Write down the values of $\alpha + \beta + \gamma$ and $\alpha\beta + \beta\gamma + \gamma\alpha$. [2]

(b) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(ii) (a) Use the substitution $x = \dfrac{1}{u}$ to find a cubic equation in u with integer coefficients. [2]

(b) Use your answer to part **(ii) (a)** to find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. [2]

(Q6, June 2007)

7.

The cubic equation $2x^3 - 3x^2 + 24x + 7 = 0$ has roots α, β and γ.

(i) Use the substitution $x = \frac{1}{u}$ to find a cubic equation in u with integer coefficients. [2]

(ii) Hence, or otherwise, find the value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$. [2]

(Q3, Jan 2008)

8.

(i) Show that $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$. [2]

(ii) The quadratic equation $x^2 - 5x + 7 = 0$ has roots α and β. Find a quadratic equation with roots α^3 and β^3. [6]

(Q9, Jan 2008)

9.

The quadratic equation $x^2 + kx + 2k = 0$, where k is a non-zero constant, has roots α and β. Find a quadratic equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. [7]

(Q8, June 2008)

10.

(i) Show that $(\alpha - \beta)^2 \equiv (\alpha + \beta)^2 - 4\alpha\beta$. **[2]**

The quadratic equation $x^2 - 6kx + k^2 = 0$, where k is a positive constant, has roots α and β, with $\alpha > \beta$.

(ii) Show that $\alpha - \beta = 4\sqrt{2}k$. **[4]**

(iii) Hence find a quadratic equation with roots $\alpha + 1$ and $\beta - 1$. **[4]**

(Q8, Jan 2009)

11.

The roots of the quadratic equation $x^2 + x - 8 = 0$ are p and q. Find the value of $p + q + \frac{1}{p} + \frac{1}{q}$. **[4]**

(Q4, June 2009)

12.

The cubic equation $x^3 + 5x^2 + 7 = 0$ has roots α, β and γ.

(i) Use the substitution $x = \sqrt{u}$ to find a cubic equation in u with integer coefficients. **[3]**

(ii) Hence find the value of $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$. **[2]**

(Q5, June 2009)

13.

The cubic equation $2x^3 + 3x - 3 = 0$ has roots α, β and γ.

(i) Use the substitution $x = u - 1$ to find a cubic equation in u with integer coefficients. **[3]**

(ii) Hence find the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$. **[2]**

(Q2, Jan 2010)

14.

The quadratic equation $x^2 + 2kx + k = 0$, where k is a non-zero constant, has roots α and β. Find a quadratic equation with roots $\dfrac{\alpha + \beta}{\alpha}$ and $\dfrac{\alpha + \beta}{\beta}$. **[7]**

(Q7, June 2010)

15.

The quadratic equation $2x^2 - x + 3 = 0$ has roots α and β, and the quadratic equation $x^2 - px + q = 0$ has roots $\alpha + \dfrac{1}{\alpha}$ and $\beta + \dfrac{1}{\beta}$.

(i) Show that $p = \frac{5}{6}$. **[4]**

(ii) Find the value of q. **[5]**

(Q8, Jan 2011)

16.

The cubic equation $x^3 + 3x^2 + 2 = 0$ has roots α, β and γ.

(i) Use the substitution $x = \dfrac{1}{\sqrt{u}}$ to show that $4u^3 + 12u^2 + 9u - 1 = 0$. **[5]**

(ii) Hence find the values of $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$ and $\dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\beta^2\gamma^2} + \dfrac{1}{\gamma^2\alpha^2}$. **[5]**

(Q10, June 2011)

17.

The cubic equation $3x^3 - 9x^2 + 6x + 2 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **[3]**

The cubic equation $x^3 + ax^2 + bx + c = 0$ has roots α^2, β^2 and γ^2.

(ii) Show that $c = -\frac{4}{9}$ and find the values of a and b. **[9]**

(Q10, Jan 2012)

18.

The quadratic equation $2x^2 + x + 5 = 0$ has roots α and β.

(i) Use the substitution $x = \dfrac{1}{u+1}$ to obtain a quadratic equation in u with integer coefficients. **[3]**

(ii) Hence, or otherwise, find the value of $\left(\dfrac{1}{\alpha} - 1\right)\left(\dfrac{1}{\beta} - 1\right)$. **[3]**

(Q6, June 2012)

19.

The quadratic equation $x^2 + x + k = 0$ has roots α and β.

(i) Use the substitution $x = 2u + 1$ to obtain a quadratic equation in u. **[2]**

(ii) Hence, or otherwise, find the value of $\left(\dfrac{\alpha - 1}{2}\right)\left(\dfrac{\beta - 1}{2}\right)$ in terms of k. **[2]**

(Q4, Jan 2013)

20.

(i) Show that $(\alpha\beta + \beta\gamma + \gamma\alpha)^2 \equiv \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$. **[3]**

(ii) It is given that α, β and γ are the roots of the cubic equation $x^3 + px^2 - 4x + 3 = 0$, where p is a constant. Find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ in terms of p. **[5]**

(Q9, Jan 2013)

21.

The cubic equation $kx^3 + 6x^2 + x - 3 = 0$, where k is a non-zero constant, has roots α, β and γ.

Find the value of $(\alpha + 1)(\beta + 1) + (\beta + 1)(\gamma + 1) + (\gamma + 1)(\alpha + 1)$ in terms of k. **[6]**

(Q8, June 2013)

22.

The cubic equation $2x^3 + 3x + 3 = 0$ has roots α, β and γ.

(i) Use the substitution $x = u + 2$ to find a cubic equation in u. **[3]**

(ii) Hence find the value of $\frac{1}{\alpha - 2} + \frac{1}{\beta - 2} + \frac{1}{\gamma - 2}$. **[4]**

(Q5, June 2014)

23.

The roots of the equation $x^3 - kx^2 - 2 = 0$ are α, β and γ, where α is real and β and γ are complex.

(i) Show that $k = \alpha - \frac{2}{\alpha^2}$. **[2]**

(ii) Given that $\beta = u + \mathrm{i}v$, where u and v are real, find u in terms of α. **[4]**

(iii) Find v^2 in terms of α. **[4]**

(Q9, June 2014)

24.

The cubic equation $x^3+4x+3=0$ has roots α, β and γ.

(i) Use the substitution $x=\sqrt{u}$ to obtain a cubic equation in u. **[3]**

(ii) Find the value of $\alpha^4+\beta^4+\gamma^4+\alpha\beta\gamma$. **[7]**

(Q10, June 2015)

25.

The quadratic equation $kx^2+x+k=0$ has roots α and β.

(i) Write down the values of $\alpha+\beta$ and $\alpha\beta$. **[1]**

(ii) Find the value of $\left(\alpha+\frac{1}{\alpha}\right)\left(\beta+\frac{1}{\beta}\right)$ in terms of k. **[5]**

(Q3, June 2016)

26.

(i) Use an algebraic method to find the square roots of the complex number $9+40\mathrm{i}$. **[6]**

(ii) Show that $9+40\mathrm{i}$ is a root of the quadratic equation $z^2-18z+1681=0$. **[1]**

(iii) By using the substitution $z=\frac{1}{u^2}$, find the roots of the equation $1681u^4-18u^2+1=0$. Give your answers in the form $x+\mathrm{i}y$, where x and y are real. **[4]**

(Q10, June 2016)

27.

In the cubic equation $4z^3+az^2+bz+c=0$, a, b and c are real numbers. One root is $1+\frac{3}{2}\mathrm{i}$ and the sum of the roots is 6. Find the values of a, b and c. **[7]**

(Q8, June 2017)