OCR Core Maths 4

Past paper questions Parametric Equations

Edited by K V Kumaran

Email: kvkumaran@gmail.com

Phone: 07961319548

Parametric Equations

- A parametric equation is one where

$$x = f(\text{some parameter}) \quad \text{and} \quad y = g(\text{some parameter}).$$

The parameter in a set of parametric equations can be any letter, but usually either t or θ. As the parameter varies it sketches out a curve. If no restriction is given, assume the parameter varies $-\infty < t < \infty$. However the parameter can be restricted in any way, defined by an inequality on the parameter. Standard examples: $0 \leqslant \theta < 2\pi$ or $-\pi < \theta \leqslant \pi$.

- You must be able to convert a parametric curve to Cartesian form. Sometimes this is just obvious; isolate t from one of the equations and put into the other. For example

$$\begin{matrix} x = 2t \\ y = \frac{t}{t+1} \end{matrix} \quad \Rightarrow \quad t = \frac{x}{2} \quad \Rightarrow \quad y = \frac{\frac{x}{2}}{\frac{x}{2}+1} = \frac{x}{x+2}.$$

If one of x or y involves a "sin" and the other involves a "cos" then use $\sin^2 x + \cos^2 x = 1$:

$$\begin{matrix} x = 3\cos\theta \\ y = \sin\theta + 4 \end{matrix} \quad \Rightarrow \quad \begin{matrix} \cos^2\theta = \left(\frac{x}{3}\right)^2 \\ \sin^2\theta = (y-4)^2 \end{matrix} \quad \Rightarrow \quad \frac{x^2}{9} + (y-4)^2 = 1.$$

- To find where a line intersects a parametric curve, place the parameters (in terms of t) into the line and solve for t. For example find the points of intersection of

$$\begin{matrix} x = 2t^2 + 1 \\ y = \frac{1}{t} \end{matrix} \quad \text{and the line} \quad x + 4y = 7.$$

Replace the x and y in the line by $2t^2 + 1$ and $\frac{1}{t}$ respectively. Therefore

$$x + 4y = 7, \quad \Rightarrow \quad (2t^2+1) + 4\left(\frac{1}{t}\right) = 7, \quad \Rightarrow \quad t^3 - 3t + 2 = 0.$$

This cubic factorises to $(t-1)^2(t+2) = 0$ which gives $t = 1$ or $t = -2$ as solutions. Plugging these back into the original parametric equation we discover the two points $(3, 1)$ and $(9, -\frac{1}{2})$. [It is worth noting that the squared factor $(t-1)^2$ in the cubic implies the the line is a *tangent* to the curve at the point $(3, 1)$.]

- To differentiate a parametric curve

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

For stationary points you still equate $\frac{dy}{dx} = 0$ and solve. All other properties you are used to for normals and tangents still work.

For example find the equation of the normal to $x = 2t^3$, $y = \frac{1}{t}$ at the point $(16, \frac{1}{2})$. Firstly we need to discover the value of the parameter at the stated point: $y = \frac{1}{t} = \frac{1}{2}$ implies $t = 2$. Next differentiate and put in $t = 2$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-t^{-2}}{6t^2} = -\frac{1}{6t^4}.$$
$$\text{When } t = 2, \frac{dy}{dx} = -\frac{1}{96}.$$

Therefore the gradient of the normal is 96. Thus, $y - \frac{1}{2} = 96(x - 16)$ which 'simplifies' to $192x - 2y - 3071 = 0$.

- In harder examples questions will leave the parameter unevaluated; either leaving it as t or setting $t = p$. For example, find the equation of the tangent to the curve $x = 2t$, $y = \frac{1}{t^2}$ where $t = p$. When $t = p$, the point becomes $(2p, \frac{1}{p^2})$. Differentiating we find

$$\frac{dy}{dx} = \frac{-2t^{-3}}{2} = -\frac{1}{t^3}.$$

Therefore the gradient of the tangent when $t = p$ is $-\frac{1}{p^3}$. Therefore the tangent is

$$y - \frac{1}{p^2} = -\frac{1}{p^3}(x - 2p) \qquad \Rightarrow \qquad x + p^3y = 3p.$$

The question could further be extended to find the area of the triangle formed by the points where the tangent crosses the x-axis and y-axis and the origin. The tangent $(x+p^3y = 3p)$ crosses the x-axis when $y = 0$ which gives $x = 3p$. The tangent crosses the y-axis when $x = 0$ which gives $y = \frac{3}{p^2}$. So the three vertices of the triangle are at $(0, 0)$, $(0, \frac{3}{p^2})$ and $(3p, 0)$. The area of the triangle is therefore

$$\text{Area} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} \times 3p \times \tfrac{3}{p^2} = \tfrac{3}{2p}.$$

1.

A curve is given parametrically by the equations

$$x = t^2, \qquad y = \frac{1}{t}.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t, giving your answer in its simplest form. [3]

(ii) Show that the equation of the tangent at the point $P\left(4, -\frac{1}{2}\right)$ is

$$x - 16y = 12.$$ [3]

(iii) Find the value of the parameter at the point where the tangent at P meets the curve again. [4]

Q7 June 2005

2.

A curve is given parametrically by the equations $x = t^2$, $y = 2t$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t, giving your answer in its simplest form. [2]

(ii) Show that the equation of the tangent to the curve at $(p^2, 2p)$ is

$$py = x + p^2.$$ [2]

(iii) Find the coordinates of the point where the tangent at $(9, 6)$ meets the tangent at $(25, -10)$. [4]

Q5 Jan 2006

3.

A curve is given parametrically by the equations

$$x = 4\cos t, \qquad y = 3\sin t,$$

where $0 \leqslant t \leqslant \frac{1}{2}\pi$.

(i) Find $\frac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent at the point P, where $t = p$, is

$$3x\cos p + 4y\sin p = 12.$$ [3]

(iii) The tangent at P meets the x-axis at R and the y-axis at S. O is the origin. Show that the area of triangle ORS is $\frac{12}{\sin 2p}$. [3]

(iv) Write down the least possible value of the area of triangle ORS, and give the corresponding value of p. [3]

Q9 June 2006

4.

The parametric equations of a curve are $x = 2t^2$, $y = 4t$. Two points on the curve are $P\,(2p^2, 4p)$ and $Q\,(2q^2, 4q)$.

(i) Show that the gradient of the normal to the curve at P is $-p$. [2]

(ii) Show that the gradient of the chord joining the points P and Q is $\frac{2}{p+q}$. [2]

(iii) The chord PQ is the normal to the curve at P. Show that $p^2 + pq + 2 = 0$. [2]

(iv) The normal at the point $R\,(8, 8)$ meets the curve again at S. The normal at S meets the curve again at T. Find the coordinates of T. [4]

Q8 Jan 2007

5.

A curve C has parametric equations

$$x = \cos t, \quad y = 3 + 2\cos 2t, \quad \text{where } 0 \leqslant t \leqslant \pi.$$

(i) Express $\frac{dy}{dx}$ in terms of t and hence show that the gradient at any point on C cannot exceed 8. [4]

(ii) Show that all points on C satisfy the cartesian equation $y = 4x^2 + 1$. [3]

(iii) Sketch the curve $y = 4x^2 + 1$ and indicate on your sketch the part which represents C. [2]

Q5 June 2007

6.

The parametric equations of a curve are $x = t^3$, $y = t^2$.

(i) Show that the equation of the tangent at the point P where $t = p$ is

$$3py - 2x = p^3.$$ [4]

(ii) Given that this tangent passes through the point $(-10, 7)$, find the coordinates of each of the three possible positions of P. [5]

Q9 Jan 2008

7.

A curve has parametric equations

$$x = 9t - \ln(9t), \quad y = t^3 - \ln(t^3).$$

Show that there is only one value of t for which $\frac{dy}{dx} = 3$ and state that value. **[6]**

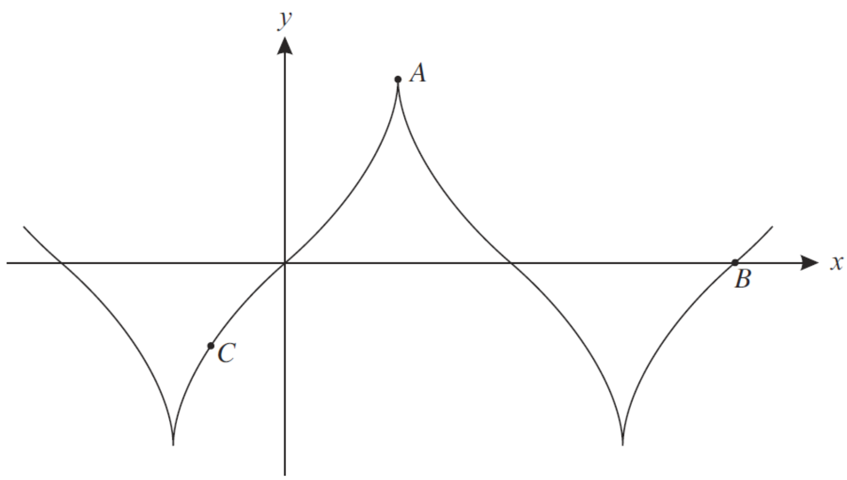
Q6 Jan 2010

8.

The parametric equations of a curve are

$$x = 2\theta + \sin 2\theta, \quad y = 4\sin\theta,$$

and part of its graph is shown below.



(i) Find the value of θ at A and the value of θ at B. [3]

(ii) Show that $\frac{dy}{dx} = \sec\theta$. [5]

(iii) At the point C on the curve, the gradient is 2. Find the coordinates of C, giving your answer in an exact form. [3]

Q9 June 2008

9.

A curve has parametric equations

$$x = t^2 - 6t + 4, \quad y = t - 3.$$

Find

(i) the coordinates of the point where the curve meets the x-axis, **[2]**

(ii) the equation of the curve in cartesian form, giving your answer in a simple form without brackets, **[2]**

(iii) the equation of the tangent to the curve at the point where $t = 2$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. **[5]**

Q6 Jan 2009

10.

A curve has parametric equations

$$x = 2t + t^2, \quad y = 2t^2 + t^3.$$

(i) Express $\frac{dy}{dx}$ in terms of t and find the gradient of the curve at the point $(3, -9)$. **[5]**

(ii) By considering $\frac{y}{x}$, find a cartesian equation of the curve, giving your answer in a form not involving fractions. **[4]**

Q5 June 2009

11.

Given that $y = \frac{\cos x}{1 - \sin x}$, find $\frac{dy}{dx}$, simplifying your answer. **[4]**

Q2 June 2010

12.

The parametric equations of a curve are $x = \frac{t+2}{t+1}$, $y = \frac{2}{t+3}$.

(i) Show that $\frac{dy}{dx} > 0$. **[6]**

(ii) Find the cartesian equation of the curve, giving your answer in a form not involving fractions. **[5]**

Q7 June 2010

13.

A curve has parametric equations

$$x = 2 + t^2, \qquad y = 4t.$$

(i) Find $\frac{dy}{dx}$ in terms of t. **[2]**

(ii) Find the equation of the normal at the point where $t = 4$, giving your answer in the form $y = mx + c$. **[3]**

(iii) Find a cartesian equation of the curve. **[2]**

Q4 Jan 2011

14.

A curve has parametric equations

$$x = \frac{1}{t+1}, \qquad y = t - 1.$$

The line $y = 3x$ intersects the curve at two points.

(i) Show that the value of t at one of these points is -2 and find the value of t at the other point. **[2]**

(ii) Find the equation of the normal to the curve at the point for which $t = -2$. **[6]**

(iii) Find the value of t at the point where this normal meets the curve again. **[2]**

(iv) Find a cartesian equation of the curve, giving your answer in the form $y = \mathrm{f}(x)$. **[3]**

Q8 June 2011

15.

A curve is defined by the parametric equations

$$x = \sin^2\theta, \quad y = 4\sin\theta - \sin^3\theta,$$

where $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{4 - 3\sin^2\theta}{2\sin\theta}$. **[3]**

(ii) Find the coordinates of the point on the curve at which the gradient is 2. **[3]**

(iii) Show that the curve has no stationary points. **[2]**

(iv) Find a cartesian equation of the curve, giving your answer in the form $y^2 = \mathrm{f}(x)$. **[2]**

Q8 Jan 2012

16.

The parametric equations of a curve are

$$x = 2 + 3\sin\theta \ \text{ and } \ y = 1 - 2\cos\theta \ \text{ for } \ 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi.$$

(i) Find the coordinates of the point on the curve where the gradient is $\frac{1}{2}$. **[5]**

(ii) Find the cartesian equation of the curve. **[2]**

Q5 Jan 2013

17.

A curve has parametric equations $x = \frac{1}{t} - 1$ and $y = 2t + \frac{1}{t^2}$.

(i) Find $\frac{dy}{dx}$ in terms of t, simplifying your answer. **[3]**

(ii) Find the coordinates of the stationary point and, by considering the gradient of the curve on either side of this point, determine its nature. **[4]**

(iii) Find a cartesian equation of the curve. **[2]**

Q9 June 2013

18.

A curve has parametric equations

$$x = 2\sin t, \quad y = \cos 2t + 2\sin t$$

for $-\frac{1}{2}\pi \leqslant t \leqslant \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = 1 - 2\sin t$ and hence find the coordinates of the stationary point. **[5]**

(ii) Find the cartesian equation of the curve. **[3]**

(iii) State the set of values that x can take and hence sketch the curve. **[3]**

Q7 June 2014

19.

The equation of a curve is $y = e^{2x}\cos x$. Find $\frac{dy}{dx}$ and hence find the coordinates of any stationary points for which $-\pi \leqslant x \leqslant \pi$. Give your answers correct to 3 significant figures. **[6]**

Q3 June 2015

20.

(i) Express $\dfrac{x+8}{x(x+2)}$ in partial fractions. **[3]**

(ii) By first using division, express $\dfrac{7x^2+16x+16}{x(x+2)}$ in the form $P+\dfrac{Q}{x}+\dfrac{R}{x+2}$. **[3]**

A curve has parametric equations $x=\dfrac{2t}{1-t}$, $y=3t+\dfrac{4}{t}$.

(iii) Show that the cartesian equation of the curve is $y=\dfrac{7x^2+16x+16}{x(x+2)}$. **[4]**

(iv) Find the area of the region bounded by the curve, the x-axis and the lines $x=1$ and $x=2$. Give your answer in the form $L+M\ln 2+N\ln 3$. **[4]**

Q10 June 2015