OCR Core Maths 3

Past paper questions
Numerical Methods

Edited by K V Kumaran

Email: kvkumaran@gmail.com

Phone: 07961319548

Numerical Methods

- Given an equation $f(x) = g(x)$ it is often not possible to solve them *analytically* (by algebraic manipulation) and we are forced to use numerical methods that 'home in' on the solution. You need to know two for C3: "search for a change of sign" and "fixed point iteration".

- *Search for a change of sign* 'homes in' on a solution to an equation by sandwiching the solution between two numbers. Those two numbers can gradually be brought together to improve knowledge of where the solution is. Given an equation ($e^x = 15x + 3$, say) it is

best to get one side equal to zero ($0 = e^x - 15x - 3$). Then *define* $f(x) = e^x - 15x - 3$. Then put values into $f(x)$ and look for a change of sign.

$$\begin{aligned} f(-1) &= 12.36787944\ldots && +\text{ ve} \\ f(0) &= -2 && -\text{ ve} \\ f(1) &= -15.28171817\ldots && -\text{ ve} \\ f(2) &= -25.6109439\ldots && -\text{ ve} \\ f(3) &= -27.91446308\ldots && -\text{ ve} \\ f(4) &= -8.401849967\ldots && -\text{ ve} \\ f(5) &= 70.4131591\ldots && +\text{ ve} \end{aligned}$$

From this we can see that there are two solutions (α and β) such that

$$-1 < \alpha < 0 \qquad \text{and} \qquad 4 < \beta < 5.$$

If you were interested in finding β to 2 decimal places (say) then you would next evaluate

$$f(4.1), f(4.2), \ldots, f(4.9)$$

and you should discover $4.1 < \beta < 4.2$. Next

$$f(4.11), f(4.12), \ldots, f(4.19)$$

and you should discover $4.18 < \beta < 4.19$. You should resist the temptation (however strong) to state $\beta = 4.18$ (to 2 d.p.) as your final answer. It is still possible that the answer could still be $\beta = 4.19$ (to 2 d.p.). You must check 4.185 and then think! *Hard*!

We find $f(4.185) < 0$, so the change of sign exists between 4.185 and 4.19 so final stated answer should be $\beta = 4.19$ (to 2 d.p.)

- *Fixed point iteration* works by taking an equation and rearranging to isolate an x in the form $x = g(x)$. From this rearrangement we form an iterative formula

$$x_{n+1} = g(x_n).$$

It is important to note that there exist many possible rearrangements of an equation; for the equation $x^3 - 3x + 4 = 0$ here are a few:

$$x = \sqrt[3]{3x-4} \qquad x = \frac{x^3+4}{3} \qquad x = \frac{3x-4}{x^2}.$$

However, the exam will usually specify which one they want[1]. In the above example let's use the first one and create $x_{n+1} = \sqrt[3]{3x_n - 4}$. The starting value for the iteration is denoted x_0 (or x_1) and you should either use the value specified in the question or choose a value close to where you know the solution exists[2]. Here let's use $x_0 = -1$.

To save time, you can use your calculator to speed up the process a lot. First type "-1 =" to enter -1 as the "Ans" on your calculator. Then type "$\sqrt[3]{(3 \times \text{Ans} - 4)}$". Press "=" repeatedly to see the results of the iteration. You should find:

$$\begin{aligned}
x_0 &= -1\\
x_1 &= -1.912931183\\
x_2 &= -2.134410543\\
x_3 &= -2.18324263\\
x_4 &= -2.19321102\\
x_5 &= -2.19528142\\
&\vdots \qquad \cdots \text{keep pressing "=" lots and eventually} \cdots\\
x &= -2.19582 \text{ to (5 d.p.)}
\end{aligned}$$

Always state the accuracy to which you give your answer (sig figs or d.p.s). If when you keep pressing "=" it settles to one number we say the iteration *converges*; otherwise it *diverges*.

- Sometimes a question gives you an iteration and asks for the equation which has been solved (or to show that the number the iteration converges to represents a solution of another given equation). All you do is remove the $n+1$ and n subscripts and rearrange: For example

$$\begin{aligned}
x_{n+1} &= \ln(\sqrt[3]{1-2x_n})\\
x &= \ln(\sqrt[3]{1-2x})\\
e^x &= \sqrt[3]{1-2x}\\
e^{3x} + 2x - 1 &= 0.
\end{aligned}$$

Simpson's Rule

- Similar to the trapezium rule is *Simpson's Rule.* It can be used to approximate integrals. It uses a quadratic curve to approximate the curve rather than a straight line, and is therefore rather more accurate. Unlike the trapezium rule it is hard to say whether the approximation will be an over or under-estimate. Therefore you don't get questions on it.

- In class I refer to "Simpson Chunks"; this is a Stone-ism you will not hear elsewhere. One "Simpson Chunk" contains two intervals/strips and three ordinates.

  In general

  $$n \text{ "Simpson Chunks"} \qquad \Leftrightarrow \qquad 2n \text{ Intervals/Strips} \qquad \Leftrightarrow \qquad 2n+1 \text{ Ordinates.}$$

  The heights on the ordinates are the y-values of the curve. They are labelled y_0, y_1, $\ldots$, y_{2n}. **Never** forget that the first height on the left is denoted y_0 and **not** y_1; if you do the whole question will go wrong because your 'odds' and 'evens' will be wrong!

- Simpson's Rule states (where h is the distance between each ordinate/height):

  $$\int_a^b y\,dx \approx \frac{h}{3}\left[y_0 + y_{2n} + 4(y_1 + y_3 + \cdots + y_{2n-1}) + 2(y_2 + y_4 + \cdots + y_{2n-2})\right].$$

  $$\int_a^b y\,dx \approx \frac{h}{3}\left[\text{'sum ends'} + 4(\text{'sum internal odds'}) + 2(\text{'sum internal evens'})\right].$$

- For example use 8 intervals to approximate $\displaystyle\int_{-4}^{4} \frac{1}{1+x^2}\,dx$. Each interval must have width 1 since the total width is 8. There must be nine ordinates. A table for the ordinates:

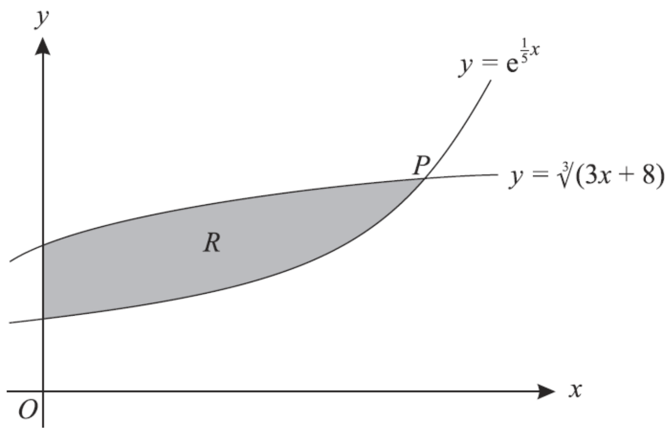
y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
$\frac{1}{17}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$

Therefore

$$\int_{-4}^{4} \frac{1}{1+x^2}\,dx \approx \frac{1}{3}\left[\frac{1}{17} + \frac{1}{17} + 4\left(\frac{1}{10} + \frac{1}{2} + \frac{1}{2} + \frac{1}{10}\right) + 2\left(\frac{1}{5} + 1 + \frac{1}{5}\right)\right]$$

$$\approx 2.573 \text{ (to 3 d.p.).}$$

1.



The diagram shows part of each of the curves $y = e^{\frac{1}{5}x}$ and $y = \sqrt[3]{(3x+8)}$. The curves meet, as shown in the diagram, at the point P. The region R, shaded in the diagram, is bounded by the two curves and by the y-axis.

(i) Show by calculation that the x-coordinate of P lies between 5.2 and 5.3. [3]

(ii) Show that the x-coordinate of P satisfies the equation $x = \frac{5}{3}\ln(3x+8)$. [2]

(iii) Use an iterative formula, based on the equation in part **(ii)**, to find the x-coordinate of P correct to 2 decimal places. [3]

(iv) Use integration, and your answer to part **(iii)**, to find an approximate value of the area of the region R. [5]

Q8 June 2005

2.

The equation $2x^3 + 4x - 35 = 0$ has one real root.

(i) Show by calculation that this real root lies between 2 and 3. [3]

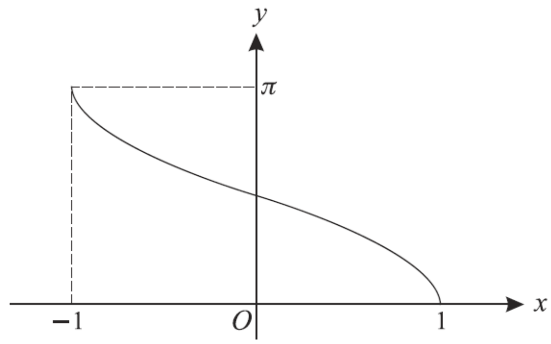
(ii) Use the iterative formula

$$x_{n+1} = \sqrt[3]{17.5 - 2x_n},$$

with a suitable starting value, to find the real root of the equation $2x^3 + 4x - 35 = 0$ correct to 2 decimal places. You should show the result of each iteration. [3]

Q3 June 2006

3.



The diagram shows the curve with equation $y = \cos^{-1} x$.

(i) Sketch the curve with equation $y = 3\cos^{-1}(x - 1)$, showing the coordinates of the points where the curve meets the axes. [3]

(ii) By drawing an appropriate straight line on your sketch in part **(i)**, show that the equation $3\cos^{-1}(x - 1) = x$ has exactly one root. [1]

(iii) Show by calculation that the root of the equation $3\cos^{-1}(x - 1) = x$ lies between 1.8 and 1.9. [2]

(iv) The sequence defined by

$$x_1 = 2, \qquad x_{n+1} = 1 + \cos\left(\tfrac{1}{3}x_n\right)$$

converges to a number α. Find the value of α correct to 2 decimal places and explain why α is the root of the equation $3\cos^{-1}(x - 1) = x$. [5]

Q7 Jan 2006

4.

(a) It is given that a and b are positive constants. By sketching graphs of

$$y = x^5 \quad \text{and} \quad y = a - bx$$

on the same diagram, show that the equation

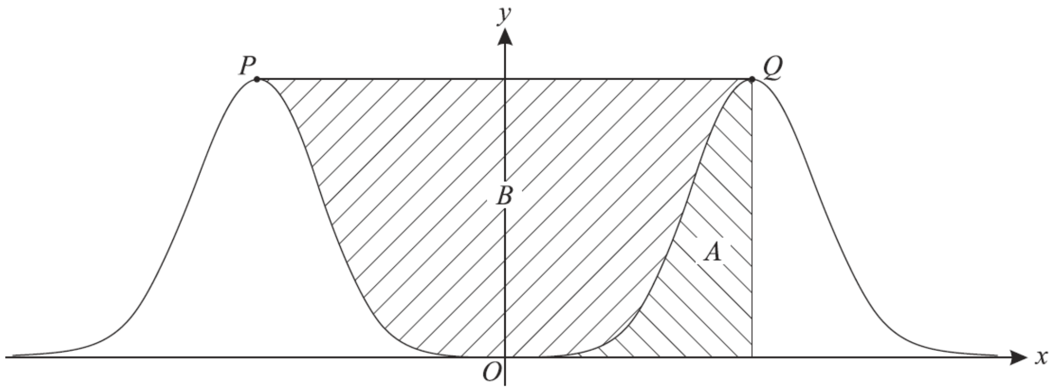
$$x^5 + bx - a = 0$$

has exactly one real root. [3]

(b) Use the iterative formula $x_{n+1} = \sqrt[5]{53 - 2x_n}$, with a suitable starting value, to find the real root of the equation $x^5 + 2x - 53 = 0$. Show the result of each iteration, and give the root correct to 3 decimal places. [4]

Q3 Jan 2007

5.



The diagram shows the curve with equation $y = x^8 e^{-x^2}$. The curve has maximum points at P and Q. The shaded region A is bounded by the curve, the line $y = 0$ and the line through Q parallel to the y-axis. The shaded region B is bounded by the curve and the line PQ.

(i) Show by differentiation that the x-coordinate of Q is 2. [5]

(ii) Use Simpson's rule with 4 strips to find an approximation to the area of region A. Give your answer correct to 3 decimal places. [4]

(iii) Deduce an approximation to the area of region B. [2]

Q8 Jan 2007

6.

The integral I is defined by

$$I = \int_0^{13} (2x+1)^{\frac{1}{3}}\,dx.$$

(i) Use integration to find the exact value of I. [4]

(ii) Use Simpson's rule with two strips to find an approximate value for I. Give your answer correct to 3 significant figures. [3]

Q4 June 2007

7.

(i) Given that $\int_0^a (6e^{2x} + x)\,dx = 42$, show that $a = \frac{1}{2}\ln\left(15 - \frac{1}{6}a^2\right)$. [5]

(ii) Use an iterative formula, based on the equation in part **(i)**, to find the value of a correct to 3 decimal places. Use a starting value of 1 and show the result of each iteration. [4]

Q6 June 2007

8.

The sequence defined by

$$x_1 = 3, \qquad x_{n+1} = \sqrt[3]{31 - \tfrac{5}{2}x_n}$$

converges to the number α.

(i) Find the value of α correct to 3 decimal places, showing the result of each iteration. [3]

(ii) Find an equation of the form $ax^3 + bx + c = 0$, where a, b and c are integers, which has α as a root. [3]

Q2 Jan 2008

9.

The definite integral I is defined by

$$I = \int_0^6 2^x \, dx.$$

(i) Use Simpson's rule with 6 strips to find an approximate value of I. [4]

(ii) By first writing 2^x in the form e^{kx}, where the constant k is to be determined, find the exact value of I. [4]

(iii) Use the answers to parts **(i)** and **(ii)** to deduce that $\ln 2 \approx \frac{9}{13}$. [2]

Q8 Jan 2008

10.

The gradient of the curve $y = (2x^2 + 9)^{\frac{5}{2}}$ at the point P is 100.

(i) Show that the x-coordinate of P satisfies the equation $x = 10(2x^2 + 9)^{-\frac{3}{2}}$. [3]

(ii) Show by calculation that the x-coordinate of P lies between 0.3 and 0.4. [3]

(iii) Use an iterative formula, based on the equation in part **(i)**, to find the x-coordinate of P correct to 4 decimal places. You should show the result of each iteration. [3]

Q4 June 2008

11.

(i) Use Simpson's rule with four strips to find an approximation to

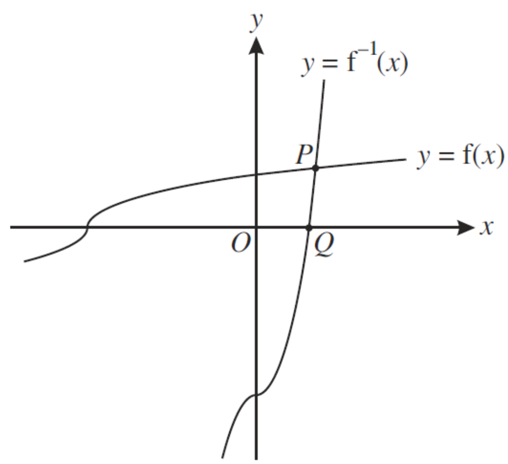
$$\int_4^{12} \ln x \, dx,$$

giving your answer correct to 2 decimal places. **[4]**

(ii) Deduce an approximation to $\int_4^{12} \ln(x^{10}) \, dx$. **[1]**

Q2 Jan 2009

12.



The function f is defined for all real values of x by

$$f(x) = \sqrt[3]{\tfrac{1}{2}x + 2}.$$

The graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at the point P, and the graph of $y = f^{-1}(x)$ meets the x-axis at Q (see diagram).

(i) Find an expression for $f^{-1}(x)$ and determine the x-coordinate of the point Q. **[3]**

(ii) State how the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are related geometrically, and hence show that the x-coordinate of the point P is the root of the equation

$$x = \sqrt[3]{\tfrac{1}{2}x + 2}.$$ **[2]**

(iii) Use an iterative process, based on the equation $x = \sqrt[3]{\tfrac{1}{2}x + 2}$, to find the x-coordinate of P, giving your answer correct to 2 decimal places. **[4]**

Q6 Jan 2009

13.

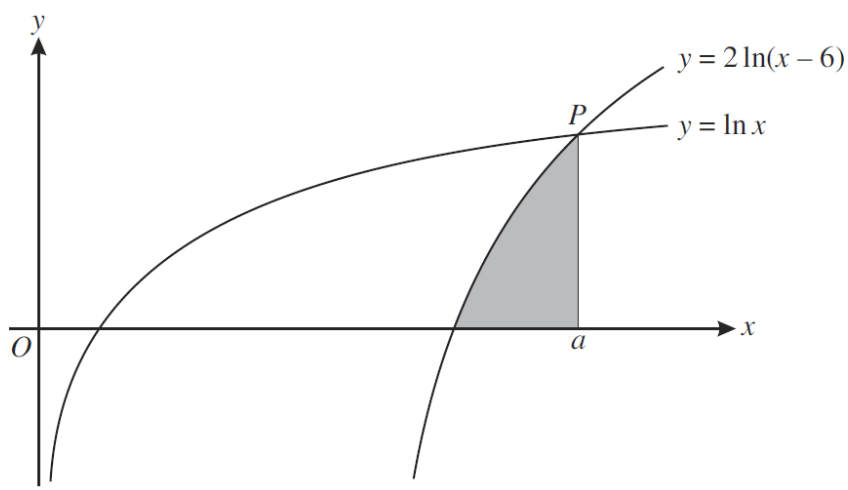
It is given that $\int_a^{3a} (e^{3x} + e^x)\,dx = 100$, where a is a positive constant.

(i) Show that $a = \frac{1}{9}\ln(300 + 3e^a - 2e^{3a})$. **[5]**

(ii) Use an iterative process, based on the equation in part **(i)**, to find the value of a correct to 4 decimal places. Use a starting value of 0.6 and show the result of each step of the process. **[4]**

Q4 June 2009

14.



The diagram shows the curves $y = \ln x$ and $y = 2\ln(x-6)$. The curves meet at the point P which has x-coordinate a. The shaded region is bounded by the curve $y = 2\ln(x-6)$ and the lines $x = a$ and $y = 0$.

(i) Give details of the pair of transformations which transforms the curve $y = \ln x$ to the curve $y = 2\ln(x-6)$. **[3]**

(ii) Solve an equation to find the value of a. **[4]**

(iii) Use Simpson's rule with two strips to find an approximation to the area of the shaded region. **[3]**

Q8 June 2009

15.

(i) Find, in simplified form, the exact value of $\int_{10}^{20} \frac{60}{x} \, dx$. **[2]**

(ii) Use Simpson's rule with two strips to find an approximation to $\int_{10}^{20} \frac{60}{x} \, dx$. **[3]**

(iii) Use your answers to parts **(i)** and **(ii)** to show that $\ln 2 \approx \frac{25}{36}$. **[2]**

Q3 Jan 2010

16.

(i) The curve $y = \sqrt{x}$ can be transformed to the curve $y = \sqrt{2x+3}$ by means of a stretch parallel to the y-axis followed by a translation. State the scale factor of the stretch and give details of the translation. **[3]**

(ii) It is given that N is a positive integer. By sketching on a single diagram the graphs of $y = \sqrt{2x+3}$ and $y = \frac{N}{x^3}$, show that the equation

$$\sqrt{2x+3} = \frac{N}{x^3}$$

has exactly one real root. **[3]**

(iii) A sequence $x_1, x_2, x_3, \ldots$ has the property that

$$x_{n+1} = N^{\frac{1}{3}}(2x_n + 3)^{-\frac{1}{6}}.$$

For certain values of x_1 and N, it is given that the sequence converges to the root of the equation $\sqrt{2x+3} = \frac{N}{x^3}$.

(a) Find the value of the integer N for which the sequence converges to the value 1.9037 (correct to 4 decimal places). **[2]**

(b) Find the value of the integer N for which, correct to 4 decimal places, $x_3 = 2.6022$ and $x_4 = 2.6282$. **[3]**

Q8 Jan 2010

17.

(i) Show by calculation that the equation

$$\tan^2 x - x - 2 = 0,$$

where x is measured in radians, has a root between 1.0 and 1.1. **[3]**

(ii) Use the iteration formula $x_{n+1} = \tan^{-1}\sqrt{2 + x_n}$ with a suitable starting value to find this root correct to 5 decimal places. You should show the outcome of each step of the process. **[4]**

(iii) Deduce a root of the equation

$$\sec^2 2x - 2x - 3 = 0.$$ **[3]**

Q6 June 2010

18.

The curve with equation $y = \dfrac{3x + 4}{x^3 - 4x^2 + 2}$ has a stationary point at P. It is given that P is close to the point with coordinates (2.4, −1.6).

(i) Find an expression for $\dfrac{dy}{dx}$ and show that the x-coordinate of P satisfies the equation

$$x = \sqrt[3]{\tfrac{16}{3}x + 1}.$$ **[4]**

(ii) By first using an iterative process based on the equation in part **(i)**, find the coordinates of P, giving each coordinate correct to 3 decimal places. **[5]**

Q6 Jan 2011

19.

The function f is defined for $x > 0$ by $f(x) = \ln x$ and the function g is defined for all real values of x by $g(x) = x^2 + 8$.

(i) Find the exact, positive value of x which satisfies the equation $fg(x) = 8$. **[3]**

(ii) State which one of f and g has an inverse and define that inverse function. **[3]**

(iii) Find the exact value of the gradient of the curve $y = gf(x)$ at the point with x-coordinate e^3. **[3]**

(iv) Use Simpson's rule with four strips to find an approximate value of

$$\int_{-4}^{4} fg(x)\, dx,$$

giving your answer correct to 3 significant figures. **[3]**

Q7 Jan 2011

20.

(i) Show by means of suitable sketch graphs that the equation

$$(x-2)^4 = x + 16$$

has exactly 2 real roots. **[3]**

(ii) State the value of the smaller root. **[1]**

(iii) Use the iterative formula

$$x_{n+1} = 2 + \sqrt[4]{x_n + 16},$$

with a suitable starting value, to find the larger root correct to 3 decimal places. **[4]**

Q4 June 2011

21.

(i) It is given that k is a positive constant. By sketching the graphs of

$$y = 14 - x^2 \text{ and } y = k\ln x$$

on a single diagram, show that the equation

$$14 - x^2 = k\ln x$$

has exactly one real root. **[3]**

(ii) The real root of the equation $14 - x^2 = 3\ln x$ is denoted by α.

(a) Find by calculation the pair of consecutive integers between which α lies. **[3]**

(b) Use the iterative formula $x_{n+1} = \sqrt{14 - 3\ln x_n}$, with a suitable starting value, to find α. Show the result of each iteration, and give α correct to 2 decimal places. **[4]**

Q5 June 2012

22.

The value of $\int_0^8 \ln(3 + x^2)\,dx$ obtained by using Simpson's rule with four strips is denoted by A.

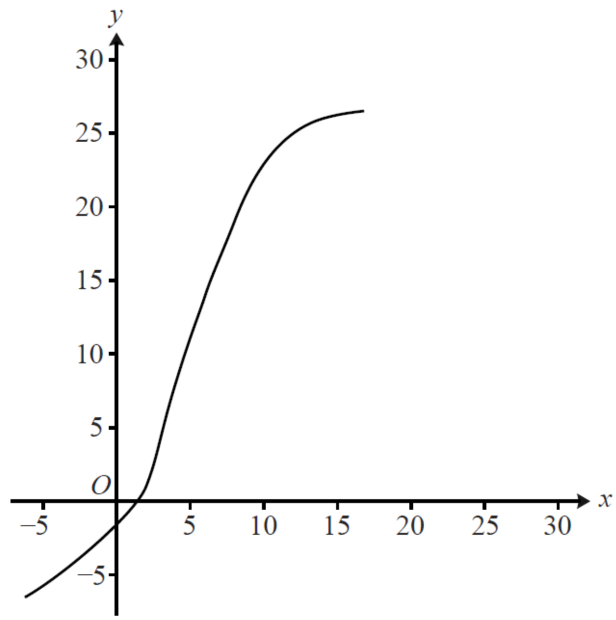
(i) Find the value of A correct to 3 significant figures. **[4]**

(ii) Explain why an approximate value of $\int_0^8 \ln(9 + 6x^2 + x^4)\,dx$ is $2A$. **[2]**

(iii) Explain why an approximate value of $\int_0^8 \ln(3e + ex^2)\,dx$ is $A+8$. **[2]**

Q6 June 2013

23.



It is given that f is a one-one function defined for all real values. The diagram shows the curve with equation $y = \mathrm{f}(x)$. The coordinates of certain points on the curve are shown in the following table.

x	2	4	6	8	10	12	14
y	1	8	14	19	23	25	26

(i) State the value of $\mathrm{ff}(6)$ and the value of $\mathrm{f}^{-1}(8)$. **[2]**

(ii) On the copy of the diagram, sketch the curve $y = \mathrm{f}^{-1}(x)$, indicating how the curves $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are related. **[2]**

(iii) Use Simpson's rule with 6 strips to find an approximation to $\int_2^{14} \mathrm{f}(x)\,\mathrm{d}x$. **[4]**

Q5 Jan 2012

24.

(i) Use Simpson's rule with four strips to find an approximation to

$$\int_0^2 \mathrm{e}^{\sqrt{x}}\,\mathrm{d}x ,$$

giving your answer correct to 3 significant figures. **[4]**

(ii) Deduce an approximation to $\int_0^2 (1 + 10\mathrm{e}^{\sqrt{x}})\,\mathrm{d}x$. **[2]**

Q3 June 2014

24.

(i) By sketching the curves $y = \ln x$ and $y = 8 - 2x^2$ on a single diagram, show that the equation

$$\ln x = 8 - 2x^2$$

has exactly one real root. **[3]**

(ii) Explain how your diagram shows that the root is between 1 and 2. **[1]**

(iii) Use the iterative formula

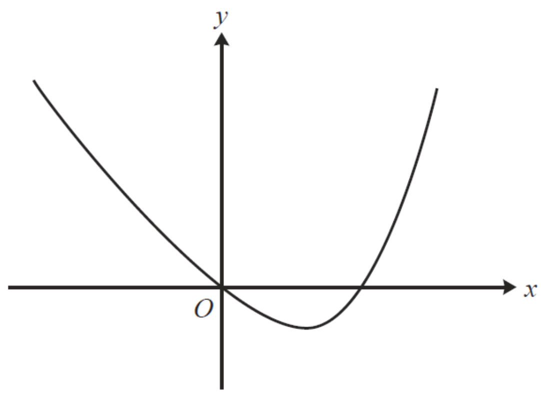
$$x_{n+1} = \sqrt{4 - \tfrac{1}{2}\ln x_n}\,,$$

with a suitable starting value, to find the root. Show all your working and give the root correct to 3 decimal places. **[4]**

(iv) The curves $y = \ln x$ and $y = 8 - 2x^2$ are each translated by 2 units in the positive x-direction and then stretched by scale factor 4 in the y-direction. Find the coordinates of the point where the new curves intersect, giving each coordinate correct to 2 decimal places. **[3]**

Q6 Jan 2013

25.



The diagram shows the curve $y = x^4 - 8x$.

(i) By sketching a second curve on the copy of the diagram, show that the equation

$$x^4 + x^2 - 8x - 9 = 0$$

has two real roots. State the equation of the second curve. **[2]**

(ii) The larger root of the equation $x^4 + x^2 - 8x - 9 = 0$ is denoted by α.

(a) Show by calculation that $2.1 < \alpha < 2.2$. **[2]**

(b) Use an iterative process based on the equation

$$x = \sqrt[4]{9 + 8x - x^2},$$

with a suitable starting value, to find α correct to 3 decimal places. Give the result of each step of the iterative process. **[4]**

Q6 June 2014

26.

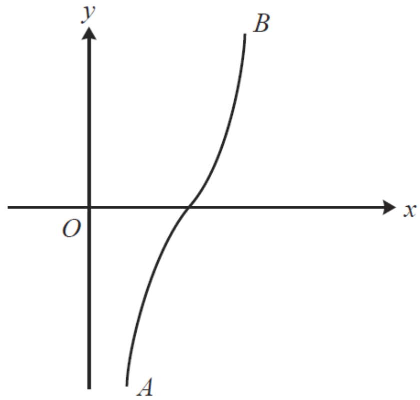
(i) Find the exact value of $\int_1^9 (7x+1)^{\frac{1}{3}} \mathrm{d}x$. **[4]**

(ii) Use Simpson's rule with two strips to show that an approximate value of $\int_1^9 (7x+1)^{\frac{1}{3}} \mathrm{d}x$ can be expressed in the form $m + n\sqrt[3]{36}$, where the values of the constants m and n are to be stated. **[3]**

(iii) Use the results from parts **(i)** and **(ii)** to find an approximate value of $\sqrt[3]{36}$, giving your answer in the form $\frac{p}{q}$ where p and q are integers. **[2]**

Q7 June 2015

27.



The diagram shows the curve $y = 8\sin^{-1}\left(x - \frac{3}{2}\right)$. The end-points A and B of the curve have coordinates $(a, -4\pi)$ and $(b, 4\pi)$ respectively.

(i) State the values of a and b. **[2]**

(ii) It is required to find the root of the equation $8\sin^{-1}\left(x - \frac{3}{2}\right) = x$.

(a) Show by calculation that the root lies between 1.7 and 1.8. **[3]**

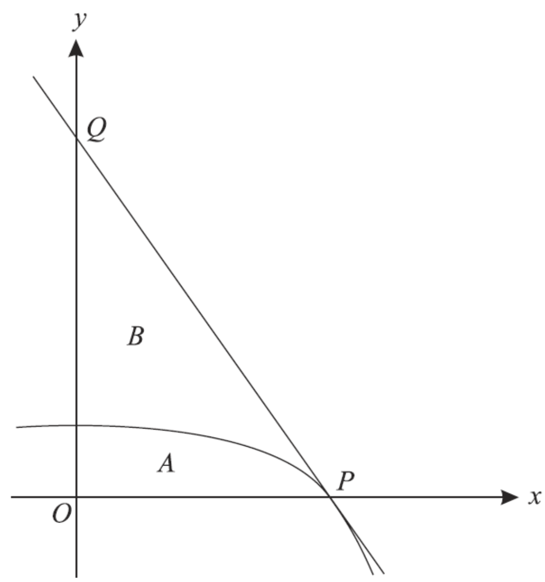
(b) In order to find the root, the iterative formula

$$x_{n+1} = p + \sin(qx_n),$$

with a suitable starting value, is to be used. Determine the values of the constants p and q and hence find the root correct to 4 significant figures. Show the result of each step of the iteration process. **[5]**

Q6 June 2015

28.



The diagram shows part of the curve $y = \ln(5 - x^2)$ which meets the x-axis at the point P with coordinates (2, 0). The tangent to the curve at P meets the y-axis at the point Q. The region A is bounded by the curve and the lines $x = 0$ and $y = 0$. The region B is bounded by the curve and the lines PQ and $x = 0$.

(i) Find the equation of the tangent to the curve at P. [5]

(ii) Use Simpson's Rule with four strips to find an approximation to the area of the region A, giving your answer correct to 3 significant figures. [4]

(iii) Deduce an approximation to the area of the region B. [2]

Q8 Jan 2006