OCR Further Pure 1

Past paper questions
Matrices

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Matrices

- Capital letters tend to be used to denote matrices and you should underline them, just as you do with vectors. An $n \times m$ matrix has n rows and m columns. So $\begin{pmatrix} 1 & 2 & -3 \\ 2 & -2 & 7 \end{pmatrix}$ is a 2×3 matrix. You must be able to add, subtract and multiply matrices. To add or subtract matrices they must be the same size and it works as you would expect. To multiply matrices ($\mathbf{A} \times \mathbf{B}$, say) the number of columns of $\mathbf{A}$ must be the same as the number of rows of $\mathbf{B}$. Your teacher will have explained this better than I ever can here, but a few examples: test for yourself!

$$\begin{pmatrix} 1 & 5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 7 & -1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 12 & 19 \\ -4 & -13 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 0 & 0 \\ 8 & 12 & -4 \end{pmatrix}.$$

- The determinant of an $n \times n$ ('*square*') matrix can be denoted by the letter Δ. A matrix with $\Delta = 0$ is called a '*singular*' matrix; otherwise it is '*non-singular*'. For a 2×2 matrix $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \Delta = ad - bc$.

- The inverse of a 2×2 matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- The inverse of a matrix (if it exists) is such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

- Matrix multiplication is not, in general, commutative; i.e. $\mathbf{AB} \neq \mathbf{BA}$. Matrix multiplication is, however, associative; i.e. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. An extension of this is $(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 \ldots \mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1} \ldots \mathbf{A}_3^{-1}\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$; prove it by induction yourself if you fancy…

- You must be very careful when manipulating matrix equations because of this non-commutativity. With normal numbers we are happy with $ax = b$ giving $x = ba^{-1}$, but this is wrong in matrix-world.

$$\begin{aligned} \mathbf{AX} &= \mathbf{B} \\ \mathbf{A}^{-1}\mathbf{AX} &= \mathbf{A}^{-1}\mathbf{B} \quad \underline{\text{pre}}\text{-multiply both sides by } \mathbf{A}^{-1} \\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B}. \end{aligned}$$

Or

$$\begin{aligned} \mathbf{XA} &= \mathbf{B} \\ \mathbf{XAA}^{-1} &= \mathbf{BA}^{-1} \quad \underline{\text{post}}\text{-multiply both sides by } \mathbf{A}^{-1} \\ \mathbf{XI} &= \mathbf{BA}^{-1} \\ \mathbf{X} &= \mathbf{BA}^{-1}. \end{aligned}$$

- Know that linear simultaneous equations can be expressed by matrices:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}.$$

The system of equations has a unique solution provided $\Delta \neq 0$. If $\Delta = 0$ then there are no unique solutions: there are either an infinite set of solutions or no solutions at all depending on whether $ax + by = c$ and $dx + ey = f$ represent parallel lines (no solutions) or the same line (infinite set of solutions).

The unique solution (if it exists) is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{ae - bd} \begin{pmatrix} e & -b \\ -d & a \end{pmatrix} \begin{pmatrix} c \\ f \end{pmatrix}$.

Matrix Transformations

- Matrices can be thought of as transformations. To discover what a matrix does, consider what it does to the arbitrary point (x, y) and *think*! For example

1. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ so $(x, y) \to (x, y)$. Therefore matrix does nothing.
2. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ so $(x, y) \to (y, x)$. Therefore the x and y-coordinates get flipped, so matrix reflected in the line $y = x$.
3. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$ so $(x, y) \to (-x, y)$. Therefore matrix changes the sign of the x-coordinate, so it represents a reflection in the y-axis.
4. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$ so $(x, y) \to (y, -x)$. Draw a few sample points and we see it represents a rotation $90°$ clockwise about the origin.
5. $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$ so $(x, y) \to (4x, 4y)$. So the x and y-coordinates get multiplied by 4. Therefore an enlargement scale factor 4, centre the origin.

- You need to know the family of matrices that represent shears.

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \text{“Shear with } x\text{-axis invariant with shear constant } k\text{”}. \quad \overrightarrow{\leftarrow}$$

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} = \text{“Shear with } y\text{-axis invariant with shear constant } k\text{”}. \quad \downarrow\uparrow$$

- For combined transformations you write the matrices in the opposite order to which the transformations occur[1]. For example if we apply transformation **A** followed by transformation **B**, then the matrix for this combined transformation would be **BA**.

- If a 2×2 matrix $\mathbf{M}$ represents a transformation, then $|\det(\mathbf{M})|$ represents the *area scale factor* of the transformation.

  For example $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement with length scale factor 2. We see the determinant is 4, so areas get multiplied by 4 in the transformation, which is consistent.

- If we consider an arbitrary matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on the point $(1, 0)$ we find

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

  (i.e. the first column of the matrix). Similarly if we act on the point $(0, 1)$ we find

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

  (i.e. the second column of the matrix). This immensely powerful pair of statements tells us that *if* a transformation can be expressed by a matrix, then all we need to do to find the matrix that does what we want is to find where $(1, 0)$ maps to under the transformation and write this image point as the first column of our matrix and find where $(0, 1)$ maps to under the transformation and write this as the second column.

- For example find the matrix that:
  - reflects in the x-axis. $(1, 0) \to (1, 0)$ and $(0, 1) \to (0, -1)$ $\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
  - reflects in the $y = x$. $(1, 0) \to (0, 1)$ and $(0, 1) \to (1, 0)$ $\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
  - rotates $90°$ clockwise. $(1, 0) \to (0, -1)$ and $(0, 1) \to (1, 0)$ $\Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

[1] Just like functions: If you apply f then g, we do $gf(x)$.

– enlarges scale factor 3. $(1,0) \to (3,0)$ and $(0,1) \to (0,3)$ $\Rightarrow \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

– stretch factor 2 parallel to y-axis. $(1,0) \to (1,0)$ and $(0,1) \to (0,2)$ $\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

– rotates $\theta°$ anticlockwise. $(1,0) \to (\cos\theta, \sin\theta)$ and $(0,1) \to (-\sin\theta, \cos\theta)$

$$\Rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

– rotates 90° CW and then reflects in y-axis. $(1,0) \to (0,-1)$ and $(0,1) \to (1,0)$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

3×3 Matrices

- To calculate the determinant of a 3×3 matrix, you pick a column or a row (most students choose the first column, but it works with any row or column) and you work down/across it using the plus/minus checkerboard approach and multiplying by the determinant of the 2×2 matrix left when the column and row of the number you have chosen is crossed out.

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \Delta = a(ei - fh) - d(bi - ch) + g(bf - ce).$$

- To invert a 3×3 matrix you do:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} +(ei-hf) & -(di-gf) & +(dh-eg) \\ -(bi-ch) & +(ai-cg) & -(ah-bg) \\ +(bf-ce) & -(af-cd) & +(ae-bd) \end{pmatrix}^{\mathrm{T}},$$
$$= \frac{1}{\Delta} \begin{pmatrix} +(ei-hf) & -(bi-ch) & +(bf-ce) \\ -(di-gf) & +(ai-cg) & -(af-cd) \\ +(dh-eg) & -(ah-bg) & +(ae-bd) \end{pmatrix}.$$

  Don't forget to transpose at the end! There is an elegant pattern to all of the above; it's easy to do once you get into the swing of it. (Mr Stone has a spreadsheet where you can practice this to your heart's content.)

- As before, a system of linear simultaneous equations can be written with a matrix.

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d \\ h \\ l \end{pmatrix}.$$

  If the matrix in non-singular then the equations have a unique solution. If the matrix is singular then the system either has an infinite set of solutions or no solutions at all. If the equations generate an inconsistency (e.g. 4=19) then there are no solutions at all.

- If the matrix is non-singular, then the unique solution is given by:

$$\mathbf{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

1.

The matrices **A** and **I** are given by $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ respectively.

(i) Find $\mathbf{A}^2$ and verify that $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$. [4]

(ii) Hence, or otherwise, show that $\mathbf{A}^{-1} = 4\mathbf{I} - \mathbf{A}$. [2]

(June 2005, Q2)

2.

[redacted]

The matrix **B** is given by $\mathbf{B} = \begin{pmatrix} a & 1 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$.

(i) Given that **B** is singular, show that $a = -\frac{2}{3}$. [3]

(ii) Given instead that **B** is non-singular, find the inverse matrix $\mathbf{B}^{-1}$. [4]

(iii) Hence, or otherwise, solve the equations

$$\begin{aligned} -x + y + 3z &= 1, \\ 2x + y - z &= 4, \\ y + 2z &= -1. \end{aligned}$$

[3]

(June 2005, Q7)

3.

(i) Write down the matrix **C** which represents a stretch, scale factor 2, in the x-direction. [2]

(ii) The matrix **D** is given by $\mathbf{D} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. Describe fully the geometrical transformation represented by **D**. [2]

(iii) The matrix **M** represents the combined effect of the transformation represented by **C** followed by the transformation represented by **D**. Show that

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$

[2]

(iv) Prove by induction that $\mathbf{M}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}$, for all positive integers n. [6]

(June 2005, Q9)

4.

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

(i) Find the value of the determinant of $\mathbf{M}$. [3]

(ii) State, giving a brief reason, whether $\mathbf{M}$ is singular or non-singular. [1]

(Jan 2006, Q3)

5.

The matrix $\mathbf{C}$ is given by $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$.

(i) Find $\mathbf{C}^{-1}$. [2]

(ii) Given that $\mathbf{C} = \mathbf{AB}$, where $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, find $\mathbf{B}^{-1}$. [5]

(Jan 2006, Q6)

6.

[redacted]

The matrix $\mathbf{T}$ is given by $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.

(i) Draw a diagram showing the unit square and its image under the transformation represented by $\mathbf{T}$. [3]

(ii) The transformation represented by matrix $\mathbf{T}$ is equivalent to a transformation A, followed by a transformation B. Give geometrical descriptions of possible transformations A and B, and state the matrices that represent them. [6]

(Jan 2006, Q8)

7.

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

(i) Find $\mathbf{A} + 3\mathbf{B}$. [2]

(ii) Show that $\mathbf{A} - \mathbf{B} = k\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and k is a constant whose value should be stated. [2]

(June 2006, Q1)

8.

The transformation S is a shear parallel to the x-axis in which the image of the point $(1, 1)$ is the point $(0, 1)$.

(i) Draw a diagram showing the image of the unit square under S. [2]

(ii) Write down the matrix that represents S. [2]

(June 2006, Q2)

9.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(i) Find $\mathbf{A}^2$ and $\mathbf{A}^3$. [3]

(ii) Hence suggest a suitable form for the matrix $\mathbf{A}^n$. [1]

(iii) Use induction to prove that your answer to part **(ii)** is correct. [4]

(June 2006, Q7)

10.

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} a & 4 & 2 \\ 1 & a & 0 \\ 1 & 2 & 1 \end{pmatrix}$.

(i) Find, in terms of a, the determinant of $\mathbf{M}$. [3]

(ii) Hence find the values of a for which $\mathbf{M}$ is singular. [3]

(iii) State, giving a brief reason in each case, whether the simultaneous equations

$$\begin{aligned} ax + 4y + 2z &= 3a, \\ x + ay \qquad &= 1, \\ x + 2y + \ z &= 3, \end{aligned}$$

have any solutions when

(a) $a = 3$,

(b) $a = 2$.

[4]

(June 2006, Q8)

11.

The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & -1 \\ -3 & -2 \end{pmatrix}$.

(i) Given that $2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$, write down the value of a. [1]

(ii) Given instead that $\mathbf{AB} = \begin{pmatrix} 7 & -4 \\ 9 & -7 \end{pmatrix}$, find the value of a. [2]

(Jan 2007, Q1)

12.

The matrix **C** is given by $\mathbf{C} = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$.

(i) Draw a diagram showing the unit square and its image under the transformation represented by **C**. [2]

The transformation represented by **C** is equivalent to a rotation, R, followed by another transformation, S.

(ii) Describe fully the rotation R and write down the matrix that represents R. [3]

(iii) Describe fully the transformation S and write down the matrix that represents S. [4]

(Jan 2007, Q9)

13.

The matrix **D** is given by $\mathbf{D} = \begin{pmatrix} a & 2 & 0 \\ 3 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$, where $a \neq 2$.

(i) Find $\mathbf{D}^{-1}$. [7]

(ii) Hence, or otherwise, solve the equations

$$\begin{aligned} ax + 2y \quad\quad &= 3, \\ 3x + y + 2z &= 4, \\ -y + z &= 1. \end{aligned}$$

[4]

(Jan 2007, Q10)

14.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$.

(i) Find $\mathbf{A}^{-1}$. [2]

The matrix $\mathbf{B}^{-1}$ is given by $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$.

(ii) Find $(\mathbf{AB})^{-1}$. [4]

(June 2007, Q4)

15.

[redacted]

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} a & 4 & 0 \\ 0 & a & 4 \\ 2 & 3 & 1 \end{pmatrix}$.

(i) Find, in terms of a, the determinant of $\mathbf{M}$. [3]

(ii) In the case when $a = 2$, state whether $\mathbf{M}$ is singular or non-singular, justifying your answer. [2]

(iii) In the case when $a = 4$, determine whether the simultaneous equations

$$\begin{aligned} ax + 4y \quad\quad &= 6, \\ ay + 4z &= 8, \\ 2x + 3y + z &= 1, \end{aligned}$$

have any solutions. [3]

(June 2007, Q7)

16.

(i) Write down the matrix, $\mathbf{A}$, that represents an enlargement, centre (0, 0), with scale factor $\sqrt{2}$. [1]

(ii) The matrix $\mathbf{B}$ is given by $\mathbf{B} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$. Describe fully the geometrical transformation represented by $\mathbf{B}$. [3]

(iii) Given that $\mathbf{C} = \mathbf{AB}$, show that $\mathbf{C} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. [1]

(iv) Draw a diagram showing the unit square and its image under the transformation represented by $\mathbf{C}$. [2]

(v) Write down the determinant of $\mathbf{C}$ and explain briefly how this value relates to the transformation represented by $\mathbf{C}$. [2]

(June 2007, Q9)

17.

The transformation S is a shear with the y-axis invariant (i.e. a shear parallel to the y-axis). It is given that the image of the point (1, 1) is the point (1, 0).

(i) Draw a diagram showing the image of the unit square under the transformation S. [2]

(ii) Write down the matrix that represents S. [2]

(Jan 2008, Q1)

18.

The matrices **A**, **B** and **C** are given by $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{C} = (2 \quad 4 \quad -1)$. Find

(i) $\mathbf{A} - 4\mathbf{B}$, [2]

(ii) $\mathbf{BC}$, [4]

(iii) $\mathbf{CA}$. [2]

(Jan 2008, Q5)

19.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} a & 3 \\ -2 & 1 \end{pmatrix}$.

(i) Given that **A** is singular, find a. [2]

(ii) Given instead that **A** is non-singular, find $\mathbf{A}^{-1}$ and hence solve the simultaneous equations

$$\begin{aligned} ax + 3y &= 1, \\ -2x + y &= -1. \end{aligned}$$

[5]

(Jan 2008, Q7)

20.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}$ and **I** is the 2×2 identity matrix. Find

(i) $\mathbf{A} - 3\mathbf{I}$, [2]

(ii) $\mathbf{A}^{-1}$. [2]

(June 2008, Q1)

21.

Describe fully the geometrical transformation represented by each of the following matrices:

(i) $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$, [1]

(ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, [2]

(iii) $\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$, [2]

(iv) $\begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$. [2]

(June 2008, Q7)

22.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} a & 8 & 10 \\ 2 & 1 & 2 \\ 4 & 3 & 6 \end{pmatrix}$. The matrix **B** is such that $\mathbf{AB} = \begin{pmatrix} a & 6 & 1 \\ 1 & 1 & 0 \\ 1 & 3 & 0 \end{pmatrix}$.

(i) Show that **AB** is non-singular. [2]

(ii) Find $(\mathbf{AB})^{-1}$. [4]

(iii) Find $\mathbf{B}^{-1}$. [5]

(June 2008, Q10)

23.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ a & 5 \end{pmatrix}$. Find

(i) $\mathbf{A}^{-1}$, **[2]**

(ii) $2\mathbf{A} - \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$. **[2]**

(Jan 2009, Q2)

24.

Given that **A** and **B** are 2×2 non-singular matrices and **I** is the 2×2 identity matrix, simplify

$$\mathbf{B}(\mathbf{AB})^{-1}\mathbf{A} - \mathbf{I}.$$ [4]

(Jan 2009, Q4)

25.

By using the determinant of an appropriate matrix, or otherwise, find the value of k for which the simultaneous equations

$$\begin{aligned} 2x - \ \ y + \ \ z &= 7, \\ 3y + \ \ z &= 4, \\ x + ky + kz &= 5, \end{aligned}$$

do not have a unique solution for x, y and z. **[5]**

(Jan 2009, Q5)

26.

(i) The transformation P is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Give a geometrical description of transformation P. **[2]**

(ii) The transformation Q is represented by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Give a geometrical description of transformation Q. **[2]**

(iii) The transformation R is equivalent to transformation P followed by transformation Q. Find the matrix that represents R. **[2]**

(iv) Give a geometrical description of the **single** transformation that is represented by your answer to part **(iii)**. **[3]**

(Jan 2009, Q6)

27.

The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$ and **I** is the 2×2 identity matrix. Find the values of the constants a and b for which $a\mathbf{A} + b\mathbf{B} = \mathbf{I}$. **[4]**

(June 2009, Q2)

28.

The matrix **C** is given by $\mathbf{C} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$.

(i) Draw a diagram showing the image of the unit square under the transformation represented by **C**. **[3]**

The transformation represented by **C** is equivalent to a transformation S followed by another transformation T.

(ii) Given that S is a shear with the y-axis invariant in which the image of the point (1, 1) is (1, 2), write down the matrix that represents S. **[2]**

(iii) Find the matrix that represents transformation T and describe fully the transformation T. **[6]**

(June 2009, Q8)

29.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

(i) Find, in terms of a, the determinant of **A**. **[3]**

(ii) Hence find the values of a for which **A** is singular. **[3]**

(iii) State, giving a brief reason in each case, whether the simultaneous equations

$$\begin{aligned} ax + \ \ y + \ \ z &= 2a, \\ x + ay + \ \ z &= -1, \\ x + \ \ y + 2z &= -1, \end{aligned}$$

have any solutions when

(a) $a = 0$,

(b) $a = 1$.

[4]

(June 2009, Q9)

30.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} a & 2 \\ 3 & 4 \end{pmatrix}$ and **I** is the 2×2 identity matrix.

(i) Find $\mathbf{A} - 4\mathbf{I}$. **[2]**

(ii) Given that **A** is singular, find the value of a. **[3]**

(Jan 2010, Q1)

31.

(i) The transformation T is represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Give a geometrical description of T. **[2]**

(ii) The transformation T is equivalent to a reflection in the line $y = -x$ followed by another transformation S. Give a geometrical description of S and find the matrix that represents S. **[4]**

(Jan 2010, Q5)

32.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & a \end{pmatrix}$, where $a \neq 1$.

(i) Find $\mathbf{A}^{-1}$. **[7]**

(ii) Hence, or otherwise, solve the equations

$$\begin{aligned} 2x - y + z &= 1, \\ 3y + z &= 2, \\ x + y + az &= 2. \end{aligned}$$

[4]

(Jan 2010, Q9)

33.

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by $\mathbf{A} = \begin{pmatrix} 1 & -4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 0 \\ -2 & 2 \end{pmatrix}$. Find

(i) $\mathbf{AB}$, **[2]**

(ii) $\mathbf{BA} - 4\mathbf{C}$. **[4]**

(June 2010, Q2)

34.

(a) Write down the matrix that represents a reflection in the line $y = x$. **[2]**

(b) Describe fully the geometrical transformation represented by each of the following matrices:

(i) $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$, **[2]**

(ii) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$. **[2]**

(June 2010, Q5)

35.

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by $\mathbf{A} = \begin{pmatrix} 2 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. Find

(i) $2\mathbf{A} + \mathbf{B}$, **[2]**

(ii) $\mathbf{AC}$, **[2]**

(iii) $\mathbf{CB}$. **[3]**

(Jan 2011, Q1)

36.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} a & a & -1 \\ 0 & a & 2 \\ 1 & 2 & 1 \end{pmatrix}$.

(i) Find, in terms of a, the determinant of $\mathbf{A}$. **[3]**

(ii) Three simultaneous equations are shown below.

$$\begin{aligned} ax + ay - z &= -1 \\ ay + 2z &= 2a \\ x + 2y + z &= 1 \end{aligned}$$

For each of the following values of a, determine whether the equations are consistent or inconsistent. If the equations are consistent, determine whether or not there is a unique solution.

(a) $a = 0$

(b) $a = 1$

(c) $a = 2$

[6]

(June 2010, Q9)

37.

Given that $\mathbf{A}$ and $\mathbf{B}$ are non-singular square matrices, simplify

$$\mathbf{AB}(\mathbf{A}^{-1}\mathbf{B})^{-1}.$$ **[3]**

(Jan 2011, Q5)

38.

(i) Write down the matrix, $\mathbf{A}$, that represents a shear with x-axis invariant in which the image of the point (1, 1) is (4, 1). **[2]**

(ii) The matrix $\mathbf{B}$ is given by $\mathbf{B} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix}$. Describe fully the geometrical transformation represented by $\mathbf{B}$. **[2]**

(iii) The matrix $\mathbf{C}$ is given by $\mathbf{C} = \begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix}$.

(a) Draw a diagram showing the unit square and its image under the transformation represented by $\mathbf{C}$. **[3]**

(b) Write down the determinant of $\mathbf{C}$ and explain briefly how this value relates to the transformation represented by $\mathbf{C}$. **[2]**

(Jan 2011, Q7)

39.

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} a & -a & 1 \\ 3 & a & 1 \\ 4 & 2 & 1 \end{pmatrix}$.

(i) Find, in terms of a, the determinant of $\mathbf{M}$. **[3]**

(ii) Hence find the values of a for which $\mathbf{M}^{-1}$ does not exist. **[3]**

(iii) Determine whether the simultaneous equations

$$6x - 6y + z = 3k,$$
$$3x + 6y + z = 0,$$
$$4x + 2y + z = k,$$

where k is a non-zero constant, have a unique solution, no solution or an infinite number of solutions, justifying your answer. **[3]**

(Jan 2011, Q9)

40.

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & a \\ 4 & 1 \end{pmatrix}$. $\mathbf{I}$ denotes the 2×2 identity matrix. Find

(i) $\mathbf{A} + 3\mathbf{B} - 4\mathbf{I}$, **[3]**

(ii) $\mathbf{AB}$. **[2]**

(June 2011, Q1)

41.

The matrix $\mathbf{C}$ is given by $\mathbf{C} = \begin{pmatrix} a & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{pmatrix}$, where $a \neq 1$. Find $\mathbf{C}^{-1}$. **[7]**

(June 2011, Q6)

42.

The matrix $\mathbf{X}$ is given by $\mathbf{X} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$.

(i) The diagram in the printed answer book shows the unit square $OABC$. The image of the unit square under the transformation represented by $\mathbf{X}$ is $OA'B'C'$. Draw and label $OA'B'C'$. **[3]**

(ii) The transformation represented by $\mathbf{X}$ is equivalent to a transformation A, followed by a transformation B. Give geometrical descriptions of possible transformations A and B and state the matrices that represent them. **[4]**

(June 2011, Q8)

43.

The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 3 & -5 \end{pmatrix}$, and **I** is the 2×2 identity matrix. Given that $p\mathbf{A} + q\mathbf{B} = \mathbf{I}$, find the values of the constants p and q. **[5]**

(Jan 2012, Q2)

44.

(a) Find the matrix that represents a reflection in the line $y = -x$. **[2]**

(b) The matrix **C** is given by $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

(i) Describe fully the geometrical transformation represented by **C**. **[2]**

(ii) State the value of the determinant of **C** and describe briefly how this value relates to the transformation represented by **C**. **[2]**

(Jan 2012, Q5)

45.

The matrix **X** is given by $\mathbf{X} = \begin{pmatrix} a & 2 & 9 \\ 2 & a & 3 \\ 1 & 0 & -1 \end{pmatrix}$.

(i) Find the determinant of **X** in terms of a. **[3]**

(ii) Hence find the values of a for which **X** is singular. **[3]**

(iii) Given that **X** is non-singular, find $\mathbf{X}^{-1}$ in terms of a. **[4]**

(Jan 2012, Q9)

46.

The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$. Find

(i) **AB**, **[2]**

(ii) $\mathbf{B}^{-1}\mathbf{A}^{-1}$. **[3]**

(June 2012, Q2)

47.

(i) The matrix **X** is given by $\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Describe fully the geometrical transformation represented by **X**. **[2]**

(ii) The matrix **Z** is given by $\mathbf{Z} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}(2+\sqrt{3}) \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2}(1-2\sqrt{3}) \end{pmatrix}$. The transformation represented by **Z** is equivalent to the transformation represented by **X**, followed by another transformation represented by the matrix **Y**. Find **Y**. **[5]**

(iii) Describe fully the geometrical transformation represented by **Y**. **[2]**

(June 2012, Q9)

48.

The matrix **D** is given by $\mathbf{D} = \begin{pmatrix} a & 2 & -1 \\ 2 & a & 1 \\ 1 & 1 & a \end{pmatrix}$.

(i) Find the determinant of **D** in terms of a. **[3]**

(ii) Three simultaneous equations are shown below.

$$\begin{aligned} ax + 2y - z &= 0 \\ 2x + ay + z &= a \\ x + y + az &= a \end{aligned}$$

For each of the following values of a, determine whether or not there is a unique solution. If the solution is not unique, determine whether the equations are consistent or inconsistent.

(a) $a = 3$

(b) $a = 2$

(c) $a = 0$ **[7]**

(June 2012, Q10)

49.

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} a & 1 \\ 1 & 4 \end{pmatrix}$, where $a \neq \frac{1}{4}$, and **I** denotes the 2×2 identity matrix. Find

(i) $2\mathbf{A} - 3\mathbf{I}$, **[3]**

(ii) $\mathbf{A}^{-1}$. **[2]**

(Jan 2013, Q1)

50.

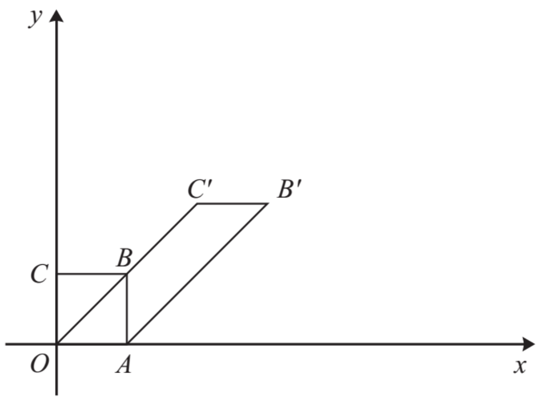
By using the determinant of an appropriate matrix, find the values of λ for which the simultaneous equations

$$\begin{aligned} 3x + 2y + 4z &= 5, \\ \lambda y + z &= 1, \\ x + \lambda y + \lambda z &= 4, \end{aligned}$$

do not have a unique solution for x, y and z. **[6]**

(Jan 2013, Q5)

51.



The diagram shows the unit square $OABC$, and its image $OAB'C'$ after a transformation. The points have the following coordinates: $A(1, 0)$, $B(1, 1)$, $C(0, 1)$, $B'(3, 2)$ and $C'(2, 2)$.

(i) Write down the matrix, **X**, for this transformation. **[2]**

(ii) The transformation represented by **X** is equivalent to a transformation P followed by a transformation Q. Give geometrical descriptions of a pair of possible transformations P and Q and state the matrices that represent them. **[6]**

(iii) Find the matrix that represents transformation Q followed by transformation P. **[2]**

(Jan 2013, Q6)

52.

The matrices **A**, **B** and **C** are given by $\mathbf{A} = (5 \quad 1)$, $\mathbf{B} = (2 \quad -5)$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

(i) Find $3\mathbf{A} - 4\mathbf{B}$. **[2]**

(ii) Find **CB**. Determine whether **CB** is singular or non-singular, giving a reason for your answer. **[5]**

(June 2013, Q2)

53.

(i) Find the matrix that represents a rotation through 90° clockwise about the origin. **[2]**

(ii) Find the matrix that represents a reflection in the x-axis. **[2]**

(iii) Hence find the matrix that represents a rotation through 90° clockwise about the origin, followed by a reflection in the x-axis. **[2]**

(iv) Describe a **single** transformation that is represented by your answer to part **(iii)**. **[2]**

(June 2013, Q7)

54.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} a & 2 & 1 \\ 1 & 3 & 2 \\ 4 & 1 & 1 \end{pmatrix}$.

(i) Find the value of a for which $\mathbf{A}$ is singular. **[5]**

(ii) Given that $\mathbf{A}$ is non-singular, find $\mathbf{A}^{-1}$ and hence solve the equations

$$\begin{aligned} ax + 2y + z &= 1, \\ x + 3y + 2z &= 2, \\ 4x + y + z &= 3. \end{aligned}$$

[7]

(June 2013, Q10)

55.

Find the determinant of the matrix $\begin{pmatrix} a & 4 & -1 \\ 3 & a & 2 \\ a & 1 & 1 \end{pmatrix}$. **[3]**

(June 2014, Q1)

56.

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -4 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{I}$ is the 2×2 identity matrix. Find

(i) $4\mathbf{A} - \mathbf{B} + 2\mathbf{I}$, **[2]**

(ii) $\mathbf{A}^{-1}$, **[2]**

(iii) $(\mathbf{AB}^{-1})^{-1}$. **[3]**

(June 2014, Q3)

57.

(a) Find the matrix that represents a shear with the y-axis invariant, the image of the point $(1, 0)$ being the point $(1, 4)$. **[2]**

(b) The matrix $\mathbf{X}$ is given by $\mathbf{X} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$.

(i) Describe fully the geometrical transformation represented by $\mathbf{X}$. **[2]**

(ii) Find the value of the determinant of $\mathbf{X}$ and describe briefly how this value relates to the transformation represented by $\mathbf{X}$. **[2]**

(June 2014, Q4)

58.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$, where a is a constant.

(i) Find $\mathbf{A}^{-1}$. **[2]**

The matrix $\mathbf{B}$ is given by $\mathbf{B} = \begin{pmatrix} 2 & a \\ 4 & 1 \end{pmatrix}$.

(ii) Given that $\mathbf{PA} = \mathbf{B}$, find the matrix $\mathbf{P}$. **[3]**

(June 2015, Q3)

59.

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$.

(i) The diagram in the Printed Answer Book shows the unit square $OABC$. The image of the unit square under the transformation represented by $\mathbf{M}$ is $OA'B'C'$. Draw and label $OA'B'C'$, indicating clearly the coordinates of A', B' and C'. **[3]**

(ii) The transformation represented by $\mathbf{M}$ is equivalent to a transformation P followed by a transformation Q. Give geometrical descriptions of a possible pair of transformations P and Q and state the matrices that represent them. **[4]**

(June 2015, Q6)

60.

The matrix $\mathbf{D}$ is given by $\mathbf{D} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & a & 3 \\ 0 & 1 & a \end{pmatrix}$.

(i) Find the values of a for which $\mathbf{D}$ is singular. **[6]**

(ii) Three simultaneous equations are shown below.

$$\begin{aligned} x + 3y + 4z &= 3 \\ 2x + ay + 3z &= 2 \\ y + az &= 0 \end{aligned}$$

For each of the following values of a, determine whether or not there is a unique solution. If a unique solution does not exist, determine whether the equations are consistent or inconsistent.

(a) $a = 3$

(b) $a = 1$

[4]

(June 2015, Q9)