Edexcel

Pure Mathematics

Year 2

Integration

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Standard Integrals

Function (f(x))	Integral $\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)}+c$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert+c$
e^x	$e^x + c$
Sin x	-Cos x + c
Cos x	Sin x + c
Tan x	ln|(Sec x)| + c
Cot x	ln|(Sin x)| + c
Cosec x	-ln|(Cosec x + Cot x)| + c
Sec x	ln|(Sec x + Tan x)| + c
Sec^2x	Tanx + c
Sec x Tan x	Sec x + c
$-Cosec^2x$	Cot x + c
e^{ax+b}	$\frac{e^{ax+b}}{a}+c$
Cos (ax + b)	$\frac{1}{a}Sin(ax+b)+c$
Sin^nxCosx	$\left(\frac{1}{n+1}\right)Sin^{n+1}x+c$
Cos^nxSinx	$\left(\frac{-1}{n+1}\right)Cos^{n+1}x+c$
Sin^2x	$\frac{1}{2}\left[x-\frac{Sin2x}{2}\right]+c$
Cos^2x	$\frac{1}{2}\left[\frac{Sin2x}{2}+x\right]+c$

Example 1

By using the substitution u = sinx or otherwise, find

$$\int 8\sin^2 x \sin 2x dx$$

Giving your answer in terms of x.

From C3 you should remember that:

$$\sin 2x = 2\sin x \cos x$$

Hence:

$$\int 8\sin^2 x \sin 2x dx = \int 16\sin^3 x \cos x dx$$

If u = sinx du = cosdx (part of the integral!)

Therefore $\int 16\sin^3 x \cos x dx = \int 16u^3 du$

$$= 4u^4 + c = 4\sin^4 x + c$$

Example 2
Use the substitution x = sin θ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{4}{(1-x^2)^{\frac{3}{2}}}dx$$

If x = sin θ then the denominator becomes (1 – $\sin^2\theta$) which is $\cos^2\theta$. When this is raised to the power of 1.5 the result is $\cos^3\theta$ as a denominator. **The x's have now been substituted but you can't forget about the dx.**
If x = sin θ then dx = cosθdθ. Therefore the integral becomes:

$$\int_0^{\frac{1}{2}} \frac{4}{(1-x^2)^{\frac{3}{2}}}dx = \int \frac{4}{(1-\sin^2\theta)^{\frac{3}{2}}}\cos\theta d\theta$$

$$= \int \frac{4}{\cos^3\theta}\cos\theta d\theta = \int \frac{4}{\cos^2\theta}d\theta$$

The next thing to spot is that $\frac{4}{\cos^2\theta} = 4\sec^2\theta$. Therefore the question becomes:

$$\int 4\sec^2\theta d\theta = 4\tan\theta$$

The question isn't quite finished as the original limits were given in terms of x. So for the lower limit of x = 0, sin θ = 0, hence θ = 0. For the upper limit x = 0.5, so sin θ = 0.5, hence θ = $\frac{\pi}{3}$. The question asks for the **exact value** so the integral is finally.

$$\int_0^{\frac{\pi}{3}} 4\sec^2\theta d\theta = [4\tan\theta]_0^{\frac{\pi}{3}}$$

$$= \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

There are a few tricks in this question but practice makes perfect and you must remember all of your differentiation and trig work from C3 and C4.

Use ideas outlined in the examples above to carry out the next question.

Use the substitution u^2 = 1 – cos2θ and integration to find
$\int_0^{\frac{\pi}{2}} Sin2\theta\sqrt{1-Cos2\theta}d\theta$.
(You will need to use implicit differentiation when dealing with the u^2.)

Integrating by Parts.

Example 3

Use integration by parts to find the exact value of $\int_1^3 x^5 \ln x dx$

The rule for integrating by parts is:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

The thing to remember is that one part of the question becomes u and the other part becomes $\frac{dv}{dx}$. Since we can't integrate lnx, it must be u and therefore x^5 must equal $\frac{dv}{dx}$. On the right of the rule we need v and $\frac{du}{dx}$ this can be achieved by integrating and differentiating respectively.

Therefore:

$$\int_1^3 x^5 \ln x dx =$$

Let $u = \ln x$ and $\frac{dv}{dx} = x^5$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^6}{6}$$

So by substituting into the rule:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_1^3 x^5 \ln x dx = \frac{x^6}{6} - \int_1^3 \frac{1}{x} \times \frac{x^6}{6}dx$$

$$= \frac{x^6}{6} - \int_1^3 \frac{x^5}{6}dx = \left[\frac{x^6}{6} - \frac{x^6}{36}\right]_1^3$$

$$= \left[\frac{5x^6}{36}\right]_1^3 = \frac{405}{4} - \frac{5}{36}$$

$$= \frac{910}{9}$$

Example 4

Use the substitution $u^2 = 2x - 1$, or otherwise, find the exact value of

$$\int_{13}^{25} \frac{6x}{\sqrt{2x-1}}dx$$

From an earlier example you should remember that there are three parts that need dealing with, 6x, the dx and the square root in the denominator

If $u^2 = 2x - 1$ then $2udu = 2dx$ (by implicit differentiation). So $dx = udu$.

If $u^2 = 2x - 1$ then $x = 0.5u^2 + 0.5$. The integral becomes:

$$\int_{13}^{25} \frac{6x}{\sqrt{2x-1}}dx = \int \frac{\left(3u^2+3\right)}{u}udu$$

$$= \int (3u^2+3)du = \left[u^3+3u\right]$$

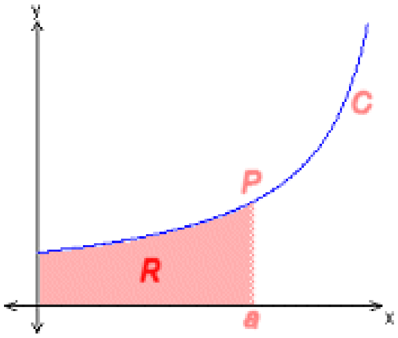
And now for the limits. If $x = 13$ then $u = 5$ and if $x = 25$ then $u = 7$.

$$\int_{13}^{25} \frac{6x}{\sqrt{2x-1}}dx = \left[u^3+3u\right]_5^7$$

$$= 224$$

Integration of Parametrics to Find Areas under curves.

Example 5



The diagram shows a sketch of the curve C with parametric equations

$$x = 7t\sin t, \quad y = 5\sec t, \quad 0 \le t < \frac{\pi}{2}$$

The point P(a, 10) lies on C.

a) Find the exact value of a.

The region R is enclosed by the curve C, the axes and the line $x = a$.

b) Show that the area of R is given by

$$35\int_0^{\frac{\pi}{3}}(\tan t + t)dt$$

c) Find the exact value of the area R.

a) Find the exact value of a.

The point P has y coordinate 10. Therefore:

$5\sec t = 10$	If $t = \frac{\pi}{3}$
$\sec t = 2$	$x = \frac{7\pi}{3}\sin\frac{\pi}{3}$
$\cos t = 0.5$	$x = \frac{7\sqrt{3}\pi}{6}$
$t = \frac{\pi}{3}$	$a = \frac{7\sqrt{3}\pi}{6}$

b) When integrating to find the area under a parametric curve you need to integrate the following:

$$\int y\frac{dx}{dt}dt$$

The x part of the parametric is a product and therefore we need to differentiate the first (7t) and multiply it second (sint) then add the first multiplied by the differential of the second. This gives:

$$\frac{dx}{dt} = 7\sin t + 7t\cos t$$

So to find the area R:

$$\int y\frac{dx}{dt}dt = \int 5\sec t(7\sin t + 7t\cos t)$$

Remembering that sect = $\frac{1}{\cos t}$ and that $\frac{\sin t}{\cos t} = \tan t$

$$35\int_0^{\frac{\pi}{3}}(\tan t + t)dt$$

The limits are defined from the work in part (a).

c) The integral from part (b) becomes:

$$35\int_0^{\frac{\pi}{3}}\left(\frac{\sin t}{\cos t} + t\right)dt$$

The integral of tant should be in your notes. The denominator nearly differentiates to give the numerator. Therefore the integral is given by the negative ln of cost.

$$35\int_0^{\frac{\pi}{3}}\left(\frac{\sin t}{\cos t} + t\right)dt = 35\int_0^{\frac{\pi}{3}}(\tan t + t)dt$$

$$= 35\left[-\ln\cos t + 0.5t^2\right]_0^{\frac{\pi}{3}}$$

= 43.5

Example 6

Use the substitution $u = 3^x$ to find the exact value of

$$\int_0^2 \frac{3^x dx}{3^x + 1}$$

The integral very quickly becomes

$$\int \frac{udx}{u + 1}$$ but we haven't dealt with the dx.

From example 9 of the C4 differentiation notes the differential of u is:

$$\frac{du}{dx} = 3^x \ln 3$$

Therefore $\frac{du}{u \ln 3} = dx$ and the integral becomes:

$$\int_1^9 \frac{du}{\ln 3(u + 1)} = \left[\frac{1}{\ln 3}\ln(u + 1)\right]_1^9$$

$$= \frac{\ln 5}{\ln 3}$$

Remember that the question said exact answer so leave it in terms of logs.

1. Find

$$\int \frac{x^2 - 5}{2x^3} \mathrm{d}x \qquad x > 0$$

giving your answer in simplest form.

(3)

(IAL P3, Jan 2021, Q1)

2. (*a*) Express $\dfrac{5x+3}{(2x-3)(x+2)}$ in partial fractions.

(3)

(*b*) Hence find the exact value of $\displaystyle\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,\mathrm{d}x$, giving your answer as a single logarithm.

(5)

(C4, June 2005 Q3)

3.

$$\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv A + \frac{B}{(2x+1)} + \frac{C}{(2x-1)}.$$

(*a*) Find the values of the constants *A*, *B* and *C*.

(4)

(*b*) Hence show that the exact value of $\displaystyle\int_1^2 \frac{2(4x^2+1)}{(2x+1)(2x-1)}\,\mathrm{d}x$ is 2 + ln *k*, giving the value of the constant *k*.

(6)

(C4, June 2007 Q4)

4.

$$\mathrm{f}(x) = \frac{4-2x}{(2x+1)(x+1)(x+3)} = \frac{A}{(2x+1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}.$$

(*a*) Find the values of the constants *A*, *B* and *C*.

(4)

(*b*) (i) Hence find $\displaystyle\int \mathrm{f}(x)\,\mathrm{d}x$.

(3)

(ii) Find $\displaystyle\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$ in the form ln *k*, where *k* is a constant.

(3)

(C4, June 2009 Q3)

5. $$f(x) = \frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}.$$

(*a*) Find the values of the constants *A*, *B* and *C*.

(4)

(*b*) (i) Hence find $\int f(x)\ dx$.

(ii) Find $\int_1^2 f(x)\ dx$, leaving your answer in the form $a + \ln b$, where a and b are constants.

(6)

(C4, June 2012 Q1)

6. Given that

$$4x^3 + 2x^2 + 17x + 8 \equiv (Ax + B)(x^2 + 4) + Cx + D$$

(*a*) find the values of the constants *A*, *B*, *C* and *D*.

(4)

(*b*) Hence find

$$\int_1^4 \frac{4x^3 + 2x^2 + 17x + 8}{x^2 + 4} dx$$

giving your answer in the form $p + \ln q$, where p and q are integers.

(6)

(C34, Jan 2014, Q3)

7. Find

(*a*) $\int (2x+3)^{12} dx$

(2)

(*b*) $\int \frac{5x}{4x^2+1} dx$

(2)

(C34, IAL June 2014, Q4)

7. Given that $k \in \mathbb{Z}^+$,

(a) show that $\int_k^{3k} \frac{2}{(3x-k)} dx$ is independent of k,

(4)

(b) show that $\int_k^{2k} \frac{2}{(2x-k)^2} dx$ is inversely proportional to k.

(3)

(New syllabus, June 2018 P1, Q7)

8. (i) Find the x coordinate of each point on the curve $y = \frac{x}{x+1}$, $x \neq -1$, at which the gradient is $\frac{1}{4}$. **(4)**

(ii) Given that

$$\int_a^{2a} \frac{t+1}{t}\,dt = \ln 7 \qquad a > 0$$

find the exact value of the constant a. **(4)**

(C34, IAL June 2015, Q5)

9. (i) Find

$$\int \left((3x+5)^9 + e^{5x} \right) dx$$

(3)

(ii) Given that b is a constant greater than 2, and

$$\int_2^b \frac{x}{x^2+5}\,dx = \ln\left(\sqrt{6}\right)$$

use integration to find the value of b.

(5)

(C34, IAL Nov 2017, Q5)

10. Use partial fractions, and integration, to find the exact value of $\int_3^4 \frac{2x^2-3}{x(x-1)}\,dx$

Write your answer in the form $a + \ln b$, where a is an integer and b is a rational constant.

(8)

(C34, IAL Nov 2017, Q8)

11. (*a*) Express $\frac{9(4+x)}{16-9x^2}$ in partial fractions.

(3)

Given that

$$f(x) = \frac{9(4+x)}{16-9x^2}, \qquad x \in \mathbb{R}, \qquad -\frac{4}{3} < x < \frac{4}{3}$$

(*b*) express $\int f(x)\,dx$ in the form $\ln(g(x))$, where $g(x)$ is a rational function.

(4)

(C34, IAL Jan 2018, Q5)

12. Given that

$$\frac{2x^2-3}{(3-2x)(1-x)^2} \equiv \frac{A}{3-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

(*a*) find the values of the constants *A*, *B* and *C*.

(4)

(*b*) Hence find

$$\int \frac{2x^2-3}{(3-2x)(1-x)^2} \, dx$$

(4)

(C34, IAL Nov 2019, Q7)

13. (i) Find, using algebraic integration, the exact value of

$$\int_3^{42} \frac{2}{3x-1} \, dx$$

giving your answer in simplest form.

(4)

(ii)
$$h(x) = \frac{2x^3-7x^2+8x+1}{(x-1)^2} \qquad x > 1$$

Given $h(x) = Ax + B + \dfrac{C}{(x-1)^2}$ where *A*, *B* and *C* are constants to be found, find

$$\int h(x) \, dx$$

(6)

(IAL P3, Jan 2020, Q8)

14. (i) Find

$$\int \frac{12}{(2x-1)^2} \, dx$$

giving your answer in simplest form.

(2)

(ii) (*a*) Write $\dfrac{4x+3}{x+2}$ in the form

$$A + \frac{B}{x+2}$$ where *A* and *B* are constants to be found

(*b*) Hence find, using algebraic integration, the exact value of

$$\int_{-8}^{-5} \frac{4x+3}{x+2} \, dx$$

giving your answer in simplest form.

(6)

(IAL P3, June 2021, Q3)

15. Find

(i) $\displaystyle\int \frac{3x-2}{3x^2-4x+5}\,dx$

(2)

(ii) $\displaystyle\int \frac{e^{2x}}{\left(e^{2x}-1\right)^3}\,dx \qquad x \neq 0$

(2)

(IAL P3, Jan 2021, Q9)

16. (*a*) Given that

$$\frac{x^2+8x-3}{x+2} \equiv Ax+B+\frac{C}{x+2} \qquad x \in \mathbb{R} \quad x \neq -2$$

find the values of the constants *A*, *B* and *C*

(3)

(*b*) Hence, using algebraic integration, find the exact value of

$$\int_0^6 \frac{x^2+8x-3}{x+2}\,dx$$

giving your answer in the form $a + b \ln 2$ where *a* and *b* are integers to be found.

(4)

(New syllabus P2, Oct 2020, Q6)

17.

Given that *k* is a positive constant,

(*a*) find

$$\int \frac{9x}{3x^2+k}\,dx$$

(2)

Given also that

$$\int_2^5 \frac{9x}{3x^2+k}\,dx = \ln 8$$

(*b*) find the value of *k*

(4)

(IAL P3, June 2022, Q3)

18.

$$f(x) = \frac{2x^3 - 4x - 15}{x^2 + 3x + 4}$$

(*a*) Show that

$$f(x) \equiv Ax + B + \frac{C(2x+3)}{x^2+3x+4}$$

where A, B and C are integers to be found.

(4)

(*b*) Hence, find

$$\int_3^5 f(x)\,dx$$

giving your answer in the form $p + \ln q$, where p and q are integers.

(5)

(IAL P3, Oct 2022, Q1)

19. (*a*) Express $\frac{3x}{(2x-1)(x-2)}$ in partial fraction form.

(3)

(*b*) Hence show that

$$\int_5^{25} \frac{3x}{(2x-1)(x-2)}\,dx = \ln k$$

where k is a fully simplified fraction to be found.

(*Solutions relying entirely on calculator technology are not acceptable*.)

(4)

(IAL P4, Oct 2022, Q2

20.

Using the substitution $u = 3 + \sqrt{2x-1}$ find the exact value of

$$\int_1^{13} \frac{4}{3+\sqrt{2x-1}}\,dx$$

giving your answer in the form $p + q \ln 2$, where p and q are integers to be found.

(8)

(IAL P4, Jan 2021, Q5)

21.

(*a*) Use the substitution $x = 2\sin u$ to show that

$$\int_0^1 \frac{3x+2}{\left(4-x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x = \int_0^p \left(\frac{3}{2}\sec u \tan u + \frac{1}{2}\sec^2 u\right)\mathrm{d}u$$

where p is a constant to be found.

(4)

(*b*) Hence find the exact value of

$$\int_0^1 \frac{3x+2}{\left(4-x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x$$

(4)

(IAL P4, June 2022, Q5)

22. Use algebraic integration and the substitution $u = \sqrt{x}$ to find the exact value of

$$\int_1^4 \frac{10}{5x+2x\sqrt{x}}\,\mathrm{d}x$$

Write your answer in the form $4\ln\left(\frac{a}{b}\right)$, where a and b are integers to be found.

(*Solutions relying entirely on calculator technology are not acceptable*.)

(8)

(IAL P4, June 2021, Q4)

23. Show that $\displaystyle\int_0^2 2x\sqrt{x+2}\ \mathrm{d}x = \frac{32}{15}(2+\sqrt{2})$.

(7)

(New syllabus, June 2018 P1, Q13)

24. (i) Using a suitable substitution, find, using calculus, the value of

$$\int_1^5 \frac{3x}{\sqrt{2x-1}}\,\mathrm{d}x$$

(6)

(ii) Find

$$\int \frac{6x^2-16}{(x+1)(2x-3)}\,\mathrm{d}x$$

(6)

(IAL P4, Oct 2020, Q7)

25. Using the substitution $u = 2x - 1$, show that

$$\int_2^5 \frac{3x+2}{2x-1}\,dx = 72 + \frac{49}{8}\ln 3$$

(6)

(C34, IAL Nov 2019, Q10)

26.

By using the substitution $u = 2x + 3$, show that

$$\int_0^{12} \frac{x}{(2x+3)^2}\,dx = \frac{1}{2}\ln 3 - \frac{2}{9}$$

(7)

27. (*a*) Use the substitution $x = u^2$, $u > 0$, to show that

$$\int \frac{1}{x(2\sqrt{x}-1)}\,dx = \int \frac{2}{u(2u-1)}\,du$$

(3)

(*b*) Hence show that

$$\int_1^9 \frac{1}{x(2\sqrt{x}-1)}\,dx = 2\ln\left(\frac{a}{b}\right)$$

where a and b are integers to be determined.

(7)

(C4, June 2013 Q5)

28.

(i) Use the substitution $u = e^x - 3$ to show that

$$\int_{\ln 5}^{\ln 7} \frac{4e^{3x}}{e^x - 3}\,dx = a + b\ln 2$$

where a and b are constants to be found.

(7)

(ii) Show, by integration, that

$$\int 3e^x \cos 2x\,dx = pe^x \sin 2x + qe^x \cos 2x + c$$

where p and q are constants to be found and c is an arbitrary constant.

(5)

(IAL P4, Oct 2022, Q7)

29. (i) Given that $y > 0$, find

$$\int \frac{3y-4}{y(3y+2)}\,dy.$$

(6)

(ii) (*a*) Use the substitution $x = 4\sin^2\theta$ to show that

$$\int_0^3 \sqrt{\left(\frac{x}{4-x}\right)}\,dx = \lambda \int_0^{\frac{\pi}{3}} \sin^2\theta \, d\theta,$$

where λ is a constant to be determined.

(5)

(*b*) Hence use integration to find

$$\int_0^3 \sqrt{\left(\frac{x}{4-x}\right)}\,dx,$$

giving your answer in the form $a\pi + b$, where a and b are exact constants.

(4)

(C4, June 2016, Q6)

30. Using the substitution $u = 2 + \sqrt{(2x + 1)}$, or other suitable substitutions, find the exact value of

$$\int_0^4 \frac{1}{2+\sqrt{(2x+1)}}\,dx$$

giving your answer in the form $A + 2\ln B$, where A is an integer and B is a positive constant.

(8)

(C4, June 2013_R Q5)

31. Using the substitution $u = \cos x + 1$, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} e^{\cos x+1}\sin x \, dx = e(e-1).$$

(6)

(C4, June 2010 Q2)

32. Using the substitution $u^2 = 2x - 1$, or otherwise, find the exact value of

$$\int_1^5 \frac{3x}{\sqrt{(2x-1)}}\,dx.$$

(8)

(C4, Jan 2006 Q3)

33. Use the substitution $x = \sin\theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx.$$

(7)

(C4, June 2005 Q4)

34. Use the substitution $u = 2^x$ to find the exact value of

$$\int_0^1 \frac{2^x}{(2^x+1)^2}\,dx.$$

(6)

(C4, June 2007 Q2)

35. Use the substitution $x = 2\sin\theta$ to find the exact value of

$$\int_0^{\sqrt{3}} \frac{1}{\left(4-x^2\right)^{\frac{3}{2}}}\,dx$$

(7)

(C34, IAL Jan 2015, Q4)

36. Using the substitution $x = 2\cos u$, or otherwise, find the exact value of

$$\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{(4-x^2)}}\,dx.$$

(7)

(C4, Jan2010 Q8)

37. (i) Find, by algebraic integration, the exact value of

$$\int_2^4 \frac{8}{(2x-3)^3}\,dx$$

(4)

(ii) Find, in simplest form,

$$\int x\left(x^2+3\right)^7 dx$$

(2)

(IAL P3, Oct 2021, Q5)

38.

Show that

$$\int_1^{e^2} x^3 \ln x\,dx = ae^8 + b$$

where a and b are rational constants to be found.

(5)

(New syllabus P1, Oct 2021, Q12)

39.

(*a*) Find $\int x \sin 2x \, dx$

(3)

(*b*) Find $\int (x + \sin 2x)^2 \, dx$

(4)

(C34, IAL Jan 2019, Q9)

40. (i) Find $\int x \sin x \, dx$

(3)

(ii) (*a*) Use the substitution $x = \sec \theta$ to show that

$$\int_1^2 \sqrt{1 - \frac{1}{x^2}} \, dx = \int_0^{\frac{\pi}{3}} \tan^2 \theta \, d\theta$$

(3)

(*b*) Hence find the exact value of

$$\int_1^2 \sqrt{1 - \frac{1}{x^2}} \, dx$$ **(4)**

(C34, IAL Oct 2018, Q8)

41. (*a*) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,dx}{(x-1)(3+2\sqrt{x-1})} = \int_p^q \frac{6\,du}{u(3+2u)}$$

where p and q are positive constants to be found. **(4)**

(*b*) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,dx}{(x-1)(3+2\sqrt{x-1})} = \ln a$$

where a is a rational constant to be found. **(6)**

(New syllabus P1, Oct 2020, Q10)

42. (i) Find

$$\int \frac{2x^2 + 5x + 1}{x^2} dx, \qquad x > 0$$

(3)

(ii) Find

$$\int x \cos 2x dx$$ **(3)**

(C34, IAL June 2018, Q1)

43. Use integration by parts to find the exact value of $\displaystyle\int_1^{e} \frac{\ln x}{x^2}\,\mathrm{d}x$

Write your answer in the form $a + \dfrac{b}{\mathrm{e}}$, where a and b are integers.

(C34, IAL June 2017, Q2)

44. Use integration by parts to find the exact value of

$$\int_0^2 x2^x\,\mathrm{d}x$$

Write your answer as a single simplified fraction.

(C34, IAL June 2016, Q5)

45. (*a*) Use integration to find

$$\int \frac{1}{x^3}\ln x\,\mathrm{d}x\,.$$

(5)

(*b*) Hence calculate

$$\int_1^2 \frac{1}{x^3}\ln x\,\mathrm{d}x\,.$$

(2)

(C4, Jan 2013 Q2)

46. (*a*) Find $\int x^2\mathrm{e}^x\,\mathrm{d}x$.

(5)

(*b*) Hence find the exact value of $\int_0^1 x^2\mathrm{e}^x\,\mathrm{d}x$.

(2)

(C4, June 2013 Q1)

47. (*a*) Show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x$$

(4)

(*b*) Hence find, using algebraic integration,

$$\int_0^{\frac{\pi}{3}} \sin^3 x\,\mathrm{d}x$$

(4)

(IAL P3, Oct 2020, Q5)

48.. (i) Find, in simplest form,

$$\int (2x-5)^7 \, dx$$

(2)

(ii) Show, by algebraic integration, that

$$\int_0^{\frac{\pi}{3}} \frac{4\sin x}{1+2\cos x} \, dx = \ln a$$

where a is a rational constant to be found.

(4)

(IAL P3, 2022 Jan, Q3)

49. $$f(\theta) = 4\cos^2\theta - 3\sin^2\theta$$

(*a*) Show that $f(\theta) = \frac{1}{2} + \frac{7}{2}\cos 2\theta$.

(3)

(*b*) Hence, using calculus, find the exact value of $\int_0^{\frac{\pi}{2}} \theta \, f(\theta) \; d\theta$.

(7)

(C4, June 2010 Q6)

50. Use integration to find the exact value of $\int_0^{\frac{\pi}{2}} x\sin 2x \; dx$. **(6)**

(C4, Jan 2011 Q1)

51. $$f(\theta) = 9\cos^2\theta + \sin^2\theta$$

(*a*) Show that $f(\theta) = a + b\cos 2\theta$, where a and b are integers which should be found.

(3)

(*b*) Using your answer to part (*a*) and integration by parts, find the exact value of

$$\int_0^{\frac{\pi}{2}} \theta^2 f(\theta) d\theta$$

(6)

(C34, IAL Jan 2016, Q8)

52. (*a*) Use integration by parts to find $\int x\sin 3x \; dx$.

(3)

(*b*) Using your answer to part (*a*), find $\int x^2 \cos 3x \; dx$. **(3)**

(C4, June 2011 Q3)

32. (*a*) Use integration by parts to find $\int xe^x \ \mathrm{d}x$.

(3)

(*b*) Hence find $\int x^2 e^x \ \mathrm{d}x$.

(3)

(C4, June 2008 Q2)

54. (*a*) Find $\int \tan^2 x \ \mathrm{d}x$.

(2)

(*b*) Use integration by parts to find $\int \frac{1}{x^3} \ln x \, \mathrm{d}x$.

(4)

(*c*) Use the substitution $u = 1 + e^x$ to show that

$$\int \frac{e^{3x}}{1+e^x} \, \mathrm{d}x = \frac{1}{2} e^{2x} - e^x + \ln(1 + e^x) + k,$$

where k is a constant.

(7)

(C4, Jan2009 Q6)

55. (i) Find $\int \ln\left(\frac{x}{2}\right) \mathrm{d}x$.

(4)

(ii) Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x \ \mathrm{d}x$.

(5)

(C4, Jan 2008 Q4)

56. (*a*) Find $\int x \cos 2x \ \mathrm{d}x$.

(4)

(*b*) Hence, using the identity $\cos 2x = 2\cos^2 x - 1$, deduce $\int x \cos^2 x \ \mathrm{d}x$.

(3)

(C4, June 2007 Q3)

57.

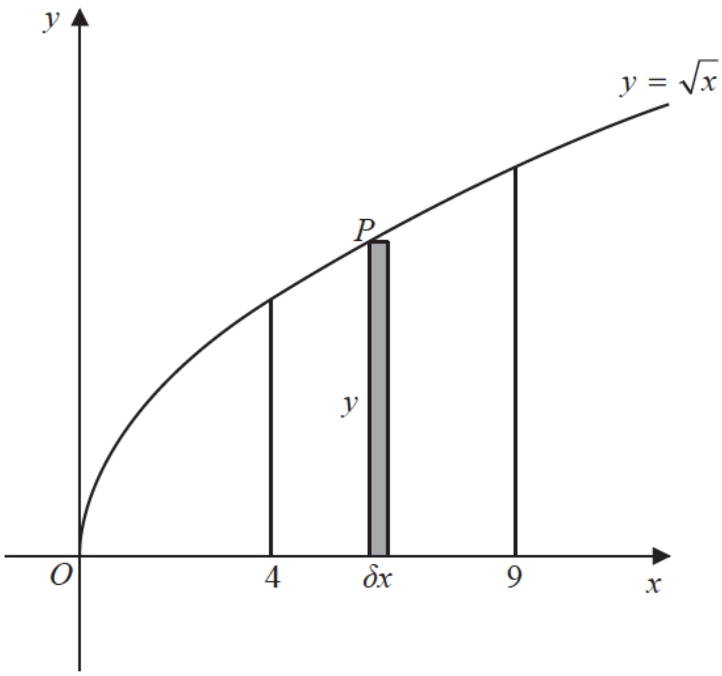


Figure 3

Figure 3 shows a sketch of the curve with equation $y = \sqrt{x}$

The point $P(x, y)$ lies on the curve.

The rectangle, shown shaded on Figure 3, has height y and width δx.

Calculate

$$\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\,\delta x$$

(3)

(New syllabus P2, June 2019, Q5)

58. (*a*) Find $\int \sqrt{(5-x)}\ \mathrm{d}x$. **(2)**

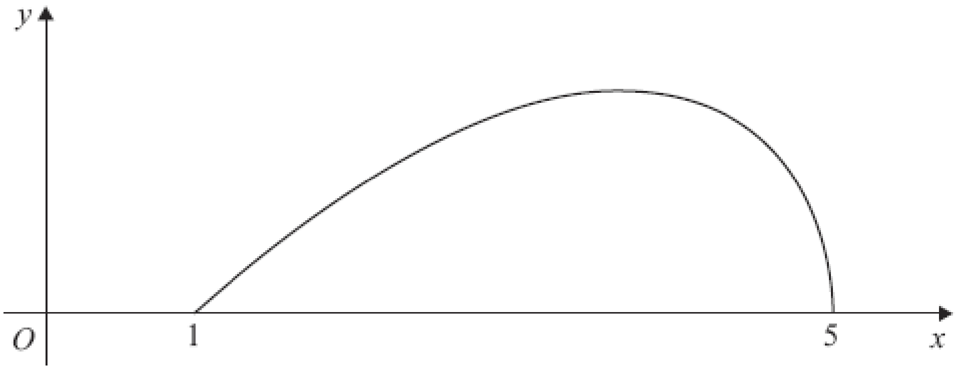


Figure 3

Figure 3 shows a sketch of the curve with equation

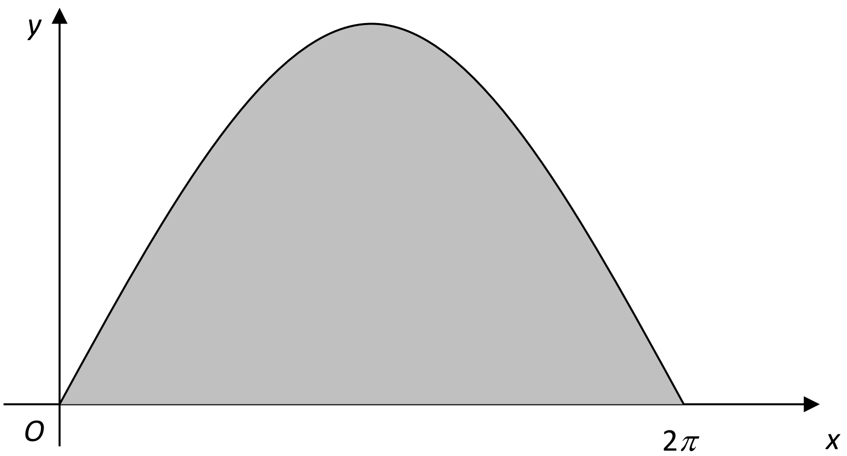
$$y = (x-1)\sqrt{(5-x)}, \quad 1 \le x \le 5$$

(*b*) (i) Using integration by parts, or otherwise, find $\int (x-1)\sqrt{(5-x)}\ \mathrm{d}x$. **(4)**

(ii) Hence find $\int_1^5 (x-1)\sqrt{(5-x)}\ \mathrm{d}x$.. **(2)**

(C4, June 2009 Q6)

59. **Figure 3**



The curve with equation $y = 3\sin\dfrac{x}{2}$, $0 \le x \le 2\pi$, is shown in Figure 1. The finite region enclosed by the curve and the x-axis is shaded.

Find, by integration, the area of the shaded region.

(3)

(C4, June 2006 Q3)

60.

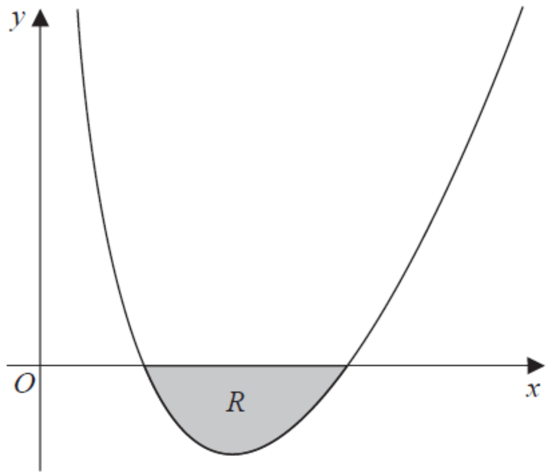


Figure 3

Figure 3 shows a sketch of part of a curve with equation

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}} \qquad x > 0$$

The region *R*, shown shaded in Figure 3, is bounded by the curve and the *x*-axis.

Find the exact area of *R*, writing your answer in the form $a\sqrt{2}+b$, where *a* and *b* are constants to be found.

(6)

(New syllabus P2, June 2022, Q8)

61. (*a*) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x-4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(*b*) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

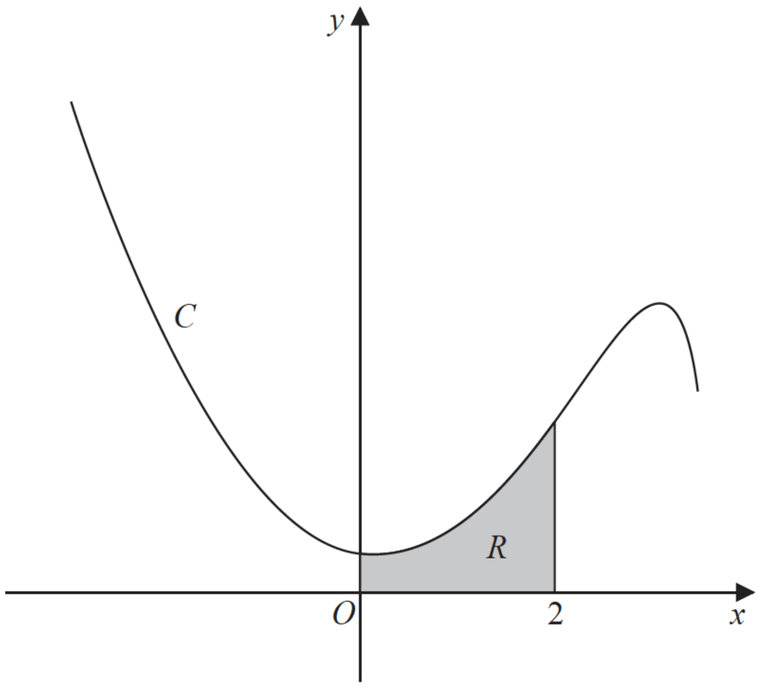


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(*c*) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

(IAL P3, Oct 2020, Q9)

62.

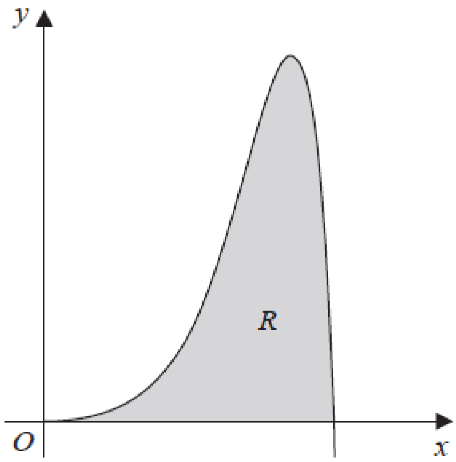


Figure 2

(*a*) Find $\int e^{2x} \sin x \, dx$

(5)

Figure 2 shows a sketch of part of the curve with equation

$$y = e^{2x} \sin x \qquad x \geq 0$$

The finite region *R* is bounded by the curve and the *x*-axis and is shown shaded in Figure 2.

(*b*) Show that the exact area of *R* is $\dfrac{e^{2\pi}+1}{5}$

(*Solutions relying on calculator technology are not acceptable*.)

(2)

(IAL P4,Jan 2021, Q7)

63.

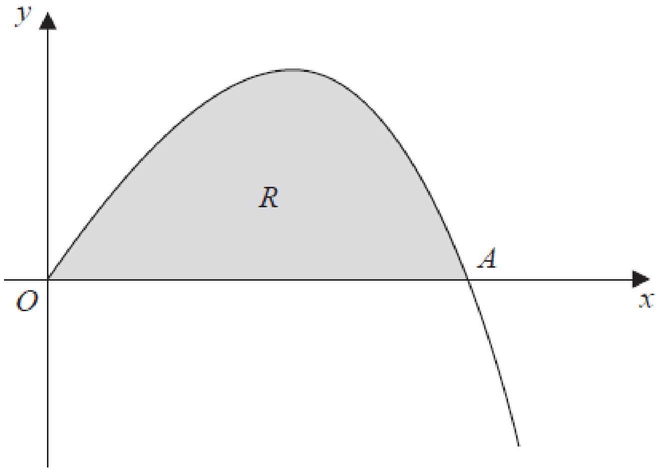


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = 4x - xe^{\frac{1}{2}x}$, $x \geq 0$.

The curve meets the x-axis at the origin O and cuts the x-axis at the point A.

(a) Find, in terms of ln 2, the x coordinate of the point A.

(2)

(b) Find $\displaystyle\int xe^{\frac{1}{2}x} \, dx$.

(3)

The finite region R, shown shaded in Figure 1, is bounded by the x-axis and the curve with equation $y = 4x - xe^{\frac{1}{2}x}$, $x \geq 0$.

(c) Find, by integration, the exact value for the area of R.
Give your answer in terms of ln 2.

(3)

(C4, June 2015, Q3)

64.

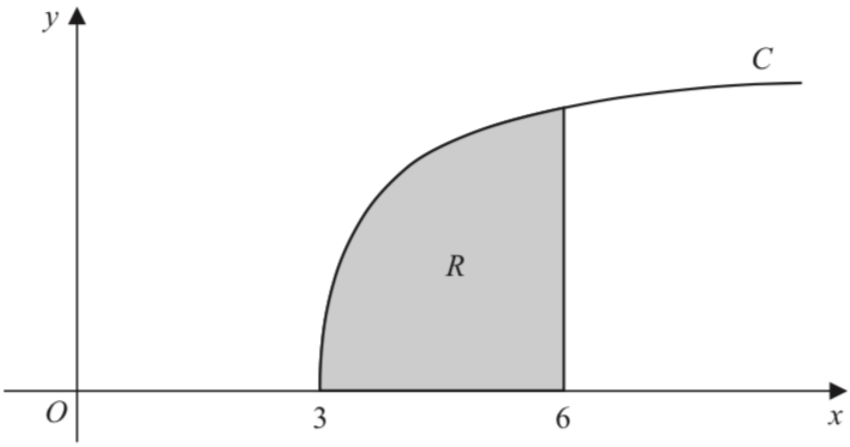


Figure 3

(*a*) By writing $\sec\theta$ as $\dfrac{1}{\cos\theta}$, show that when $x = 3\sec\theta$,

$$\frac{dx}{d\theta} = 3\sec\theta\tan\theta$$

(2)

Figure 3 shows a sketch of part of the curve C with equation

$$y = \frac{\sqrt{x^2 - 9}}{x} \qquad x \geqslant 3$$

The finite region R, shown shaded in Figure 3, is bounded by the curve C, the x-axis and the line with equation $x = 6$

(*b*) Use the substitution $x = 3\sec\theta$ to find the exact value of the area of R.
[*Solutions based entirely on graphical or numerical methods are not acceptable.*]

(7)

(C34, IAL June 2018, Q11)

65.

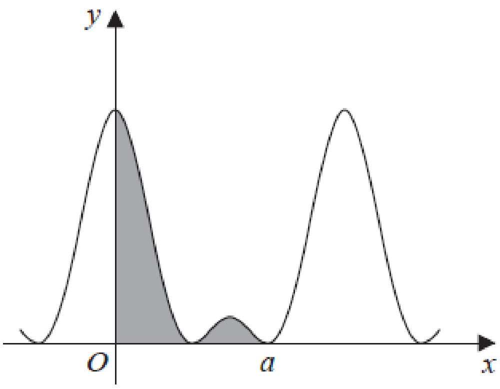


Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = (1 + 2\cos 2x)^2$$

(*a*) Show that

$$(1 + 2\cos 2x)^2 \equiv p + q\cos 2x + r\cos 4x$$

where p, q and r are constants to be found.

(2)

The curve touches the positive x-axis for the second time when $x = a$, as shown in Figure 4.

The regions bounded by the curve, the y-axis and the x-axis up to $x = a$ are shown shaded in Figure 4.

(*b*) Find, using algebraic integration and making your method clear, the exact total area of the shaded regions. Write your answer in simplest form.

(5)

(IAL P3, Oct 2021, Q10)

66.

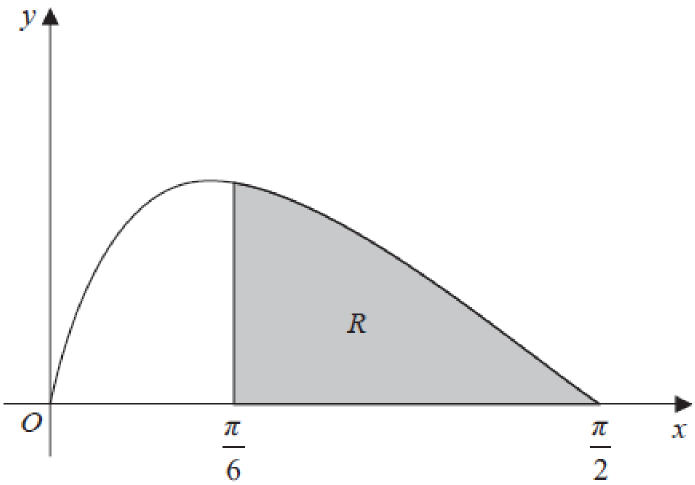


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = \frac{16\sin 2x}{(3+4\sin x)^2} \qquad 0 \leq x \leq \frac{\pi}{2}$$

The region R, shown shaded in Figure 2, is bounded by the curve, the x-axis and the line with equation $x = \frac{\pi}{6}$

Using the substitution $u = 3 + 4\sin x$, show that the area of R can be written in the form $a + \ln b$, where a and b are rational constants to be found.

(7)

(IAL P4, Oct 2021, Q6)

67.

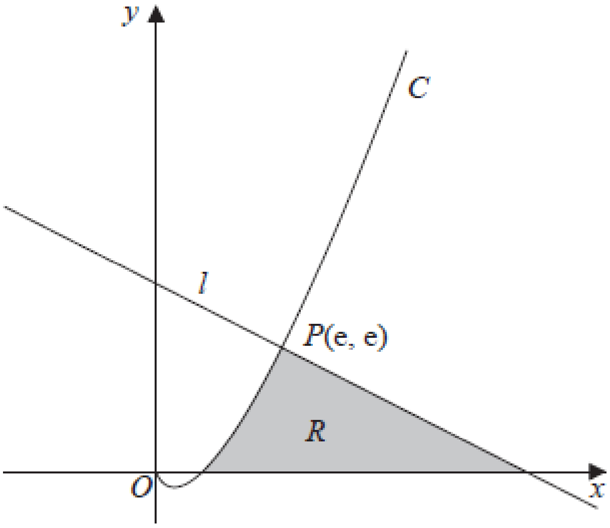


Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = x \ln x$, $x > 0$. The line l is the normal to C at the point $P(\mathrm{e}, \mathrm{e})$.

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the x-axis.

Show that the exact area of R is $A\mathrm{e}^2 + B$ where A and B are rational numbers to be found.

(10)

(New syllabus P2, June 2018, Q13)

68. The curve C with equation

$$y = \frac{p-3x}{(2x-q)(x+3)} \qquad x \in \mathbb{R}, \quad x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes
with equations $x = 2$ and $x = -3$

(*a*) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

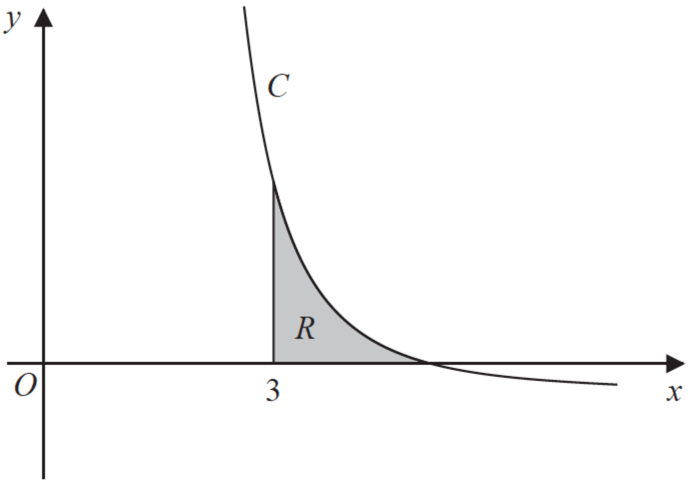


Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(*b*) Show that the exact value of the area of R is $a \ln 2 + b \ln 3$, where a and b are rational constants to be found.

(8)

(New syllabus P1, June 2019, Q13)

69.

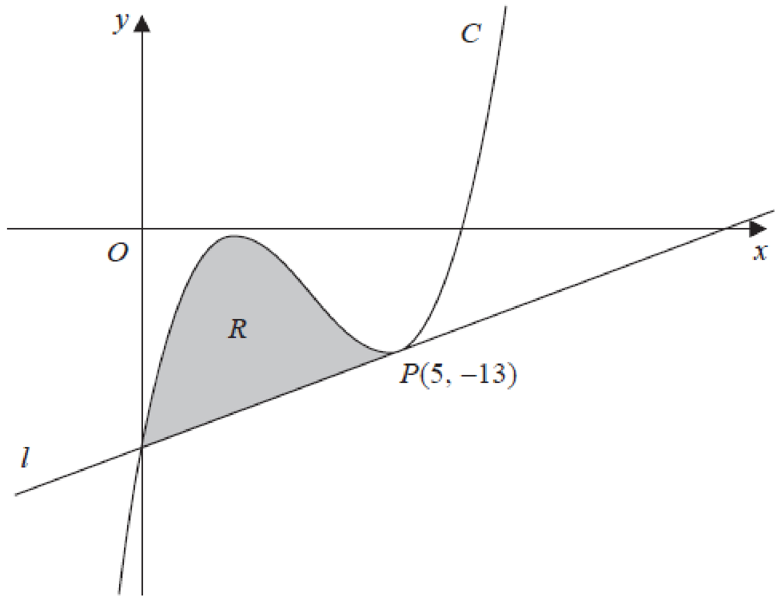


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C

The line l is the tangent to C at P

(a) Use differentiation to find the equation of l, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(4)

(b) Hence verify that l meets C again on the y-axis.

(1)

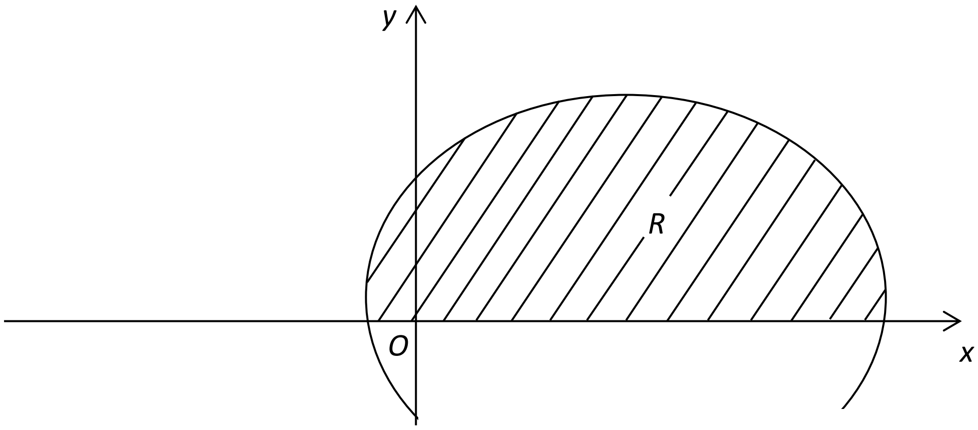
The finite region R, shown shaded in Figure 2, is bounded by the curve C and the line l.

(c) Use algebraic integration to find the exact area of R.

(4)

(New syllabus P2, Oct 2021, Q7)

70. **Figure 2**



The curve shown in Figure 2 has parametric

$$x = t - 2\sin t, \qquad y = 1 - 2\cos t, \qquad 0 \le t \le 2\pi.$$

(*a*) Show that the curve crosses the *x*-axis where $t = \dfrac{\pi}{3}$ and $t = \dfrac{5\pi}{3}$.

(2)

The finite region *R* is enclosed by the curve and the *x*-axis, as shown shaded in Figure 2.

(*b*) Show that the area *R* is given by the integral

$$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \;\; \mathrm{d}t\,.$$

(3)

(*c*) Use this integral to find the exact value of the shaded area.

(7)

(C4, Jan 2006 Q8)

71.

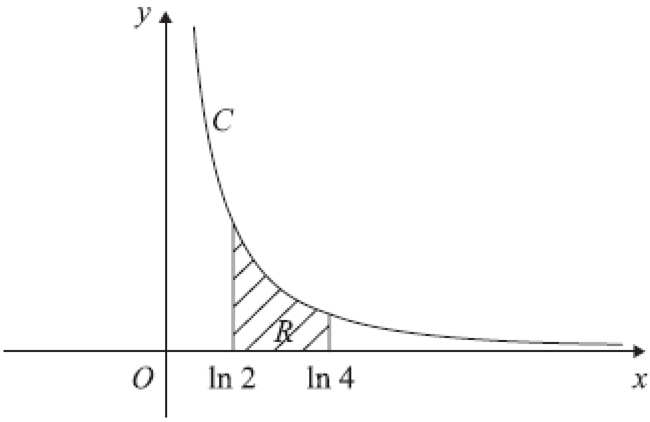


Figure 3

The curve C has parametric equations

$$x = \ln(t+2), \quad y = \frac{1}{(t+1)}, \quad t > -1.$$

The finite region R between the curve C and the x-axis, bounded by the lines with equations $x = \ln 2$ and $x = \ln 4$, is shown shaded in Figure 3.

(a) Show that the area of R is given by the integral

$$\int_0^2 \frac{1}{(t+1)(t+2)}\,\mathrm{d}t\,.$$

(4)

(b) Hence find an exact value for this area.

(6)

(c) Find a cartesian equation of the curve C, in the form $y = \mathrm{f}(x)$.

(4)

(d) State the domain of values for x for this curve.

(1)

(C4, Jan 2008 Q7)

72.

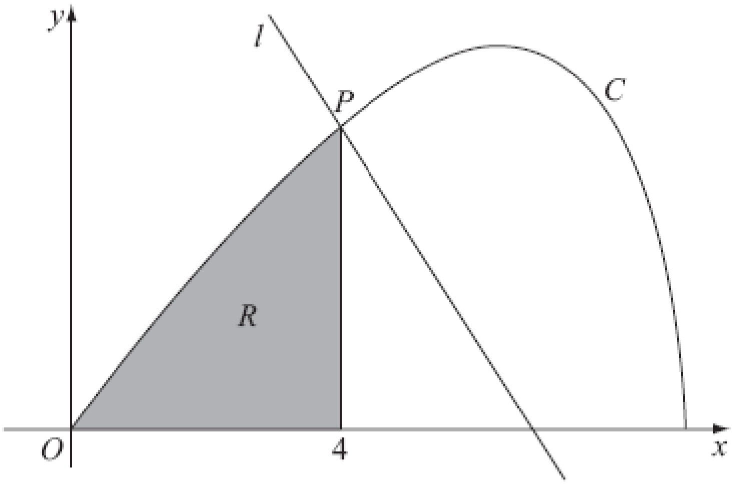


Figure 3

Figure 3 shows the curve C with parametric equations

$$x = 8\cos t, \qquad y = 4\sin 2t, \qquad 0 \le t \le \frac{\pi}{2}.$$

The point P lies on C and has coordinates $(4, 2\sqrt{3})$.

(a) Find the value of t at the point P.

(2)

The line l is a normal to C at P.

(b) Show that an equation for l is $y = -x\sqrt{3} + 6\sqrt{3}$.

(6)

The finite region R is enclosed by the curve C, the x-axis and the line $x = 4$, as shown shaded in Figure 3.

(c) Show that the area of R is given by the integral $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64\sin^2 t\cos t \, dt$.

(4)

(d) Use this integral to find the area of R, giving your answer in the form $a + b\sqrt{3}$, where a and b are constants to be determined.

(4)

(C4, June 2008 Q8)

73.

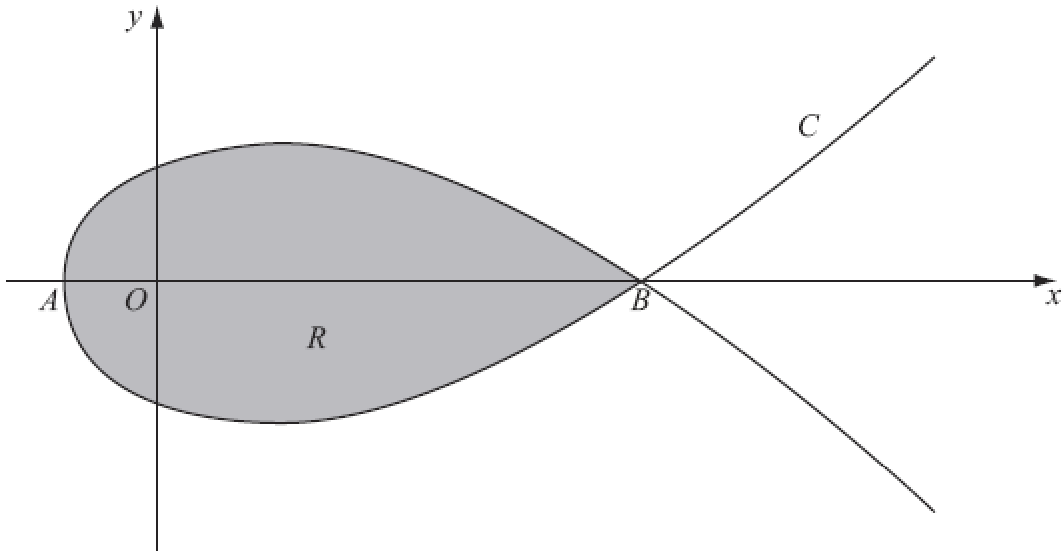


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 5t^2 - 4, \quad y = t(9 - t^2)$$

The curve C cuts the x-axis at the points A and B.

(a) Find the x-coordinate at the point A and the x-coordinate at the point B.

(3)

The region R, as shown shaded in Figure 2, is enclosed by the loop of the curve.

(b) Use integration to find the area of R.

(6)

(C4, Jan2010 Q7)

74.

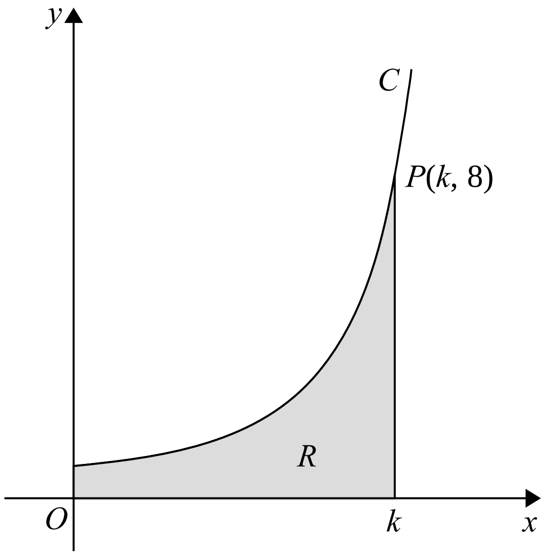


Diagram not drawn to scale

Figure 4

Figure 4 shows a sketch of part of the curve C with parametric equations

$$x = 3\theta\sin\theta, \qquad y = \sec^3\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point $P(k, 8)$ lies on C, where k is a constant.

(a) Find the exact value of k.

(2)

The finite region R, shown shaded in Figure 4, is bounded by the curve C, the y-axis, the x-axis and the line with equation $x = k$.

(b) Show that the area of R can be expressed in the form

$$\lambda\int_{\alpha}^{\beta}(\theta\sec^2\theta + \tan\theta\sec^2\theta)\,\mathrm{d}\theta$$

where λ, α and β are constants to be determined.

(4)

(c) Hence use integration to find the exact value of the area of R.

(6)

(C4, June 2017, Q8)

75.

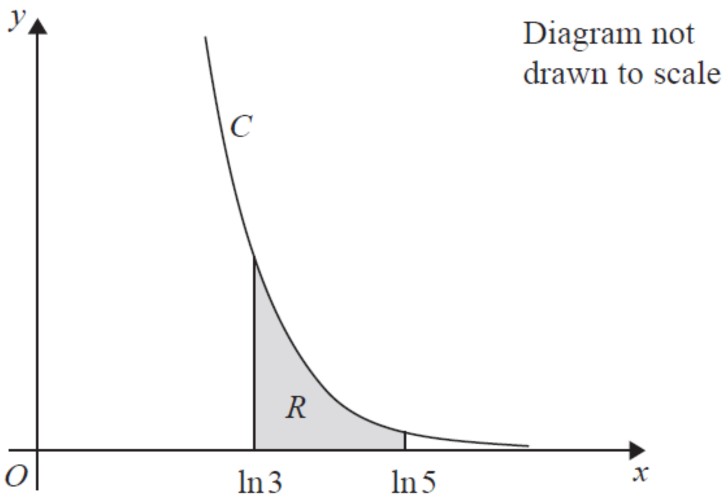


Figure 2

The curve C has parametric equations

$$x = \ln(t+2), \quad y = \frac{4}{t^2} \quad t > 0$$

The finite region R, shown shaded in Figure 2, is bounded by the curve C, the x-axis and the lines with equations $x = \ln 3$ and $x = \ln 5$.

(a) Show that the area of R is given by the integral

$$\int_1^3 \frac{4}{t^2(t+2)}\,\mathrm{d}t$$

(3)

(b) Hence find an exact value for the area of R.
Write your answer in the form $(a + \ln b)$, where a and b are rational numbers.

(7)

(c) Find a cartesian equation of the curve C in the form $y = \mathrm{f}(x)$.

(2)

(C34, IAL Jan 2015, Q9)

76.

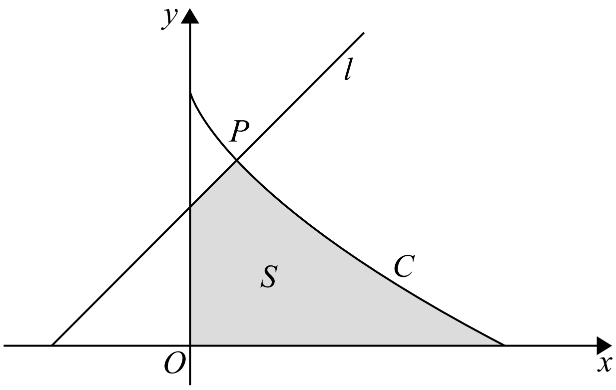


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8\cos^3\theta, \quad y = 6\sin^2\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

Given that the point P lies on C and has parameter $\theta = \frac{\pi}{3}$

(*a*) find the coordinates of P.

(2)

The line l is the normal to C at P.

(*b*) Show that an equation of l is $y = x + 3.5$

(5)

The finite region S, shown shaded in Figure 6, is bounded by the curve C, the line l, the y-axis and the x-axis.

(*c*) Show that the area of S is given by

$$4 + 144\int_0^{\frac{\pi}{3}} (\sin\theta\cos^2\theta - \sin\theta\cos^4\theta)\,\mathrm{d}\theta$$

(6)

(*d*) Hence, by integration, find the exact area of S.

(3)

(C34, IAL June 2017, Q14)

77.

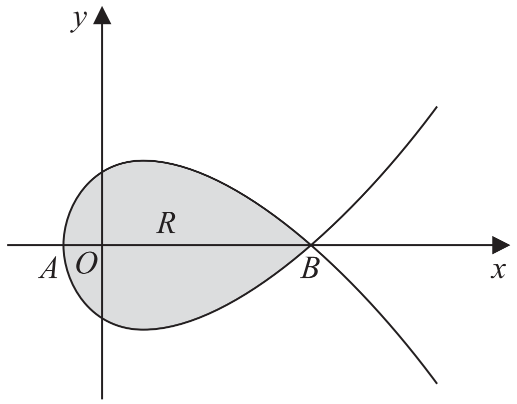


Figure 3

Figure 3 shows a sketch of part of the curve C with parametric equations

$$x = 7t^2 - 5, \qquad y = t\,(9 - t^2), \qquad t \in \mathbb{R}$$

(*a*) Find an equation of the tangent to C at the point where $t = 1$

Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(5)

The curve C cuts the x-axis at the points A and B, as shown in Figure 3

(*b*) (i) Find the x coordinate of the point A.

(ii)Find the x coordinate of the point B.

(3)

The region R, shown shaded in Figure 3, is enclosed by the loop of the curve C.

(*c*) Use integration to find the area of R.

(5)

(C34, IAL Oct 2018, Q12)

78.

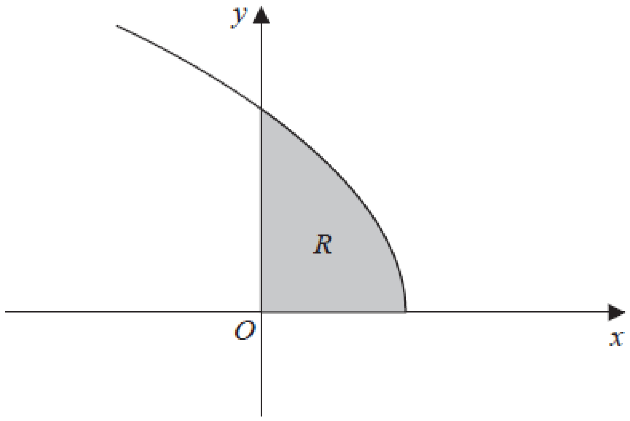


Figure 3

Figure 3 shows a sketch of the curve C with parametric equations

$$x = 2\cos 2t \qquad y = 4\sin t \qquad 0 \leq t \leq \frac{\pi}{2}$$

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and the y-axis.

(a) (i) Show, making your working clear, that the area of $R = \int_0^{\frac{\pi}{4}} 32\sin^2 t\cos t \; \mathrm{d}t$

(ii)Hence find, by algebraic integration, the exact value of the area of R.

(6)

(b) Show that all points on C satisfy $y = \sqrt{ax+b}$, where a and b are constants to be found.

(3)

The curve C has equation $y = \mathrm{f}(x)$ where f is the function

$$\mathrm{f}(x) = \sqrt{ax+b} \qquad -2 \leq x \leq 2$$

and a and b are the constants found in part (b).

(c) State the range of f.

(1)

(IAL P4, June 2021, Q6)

79.

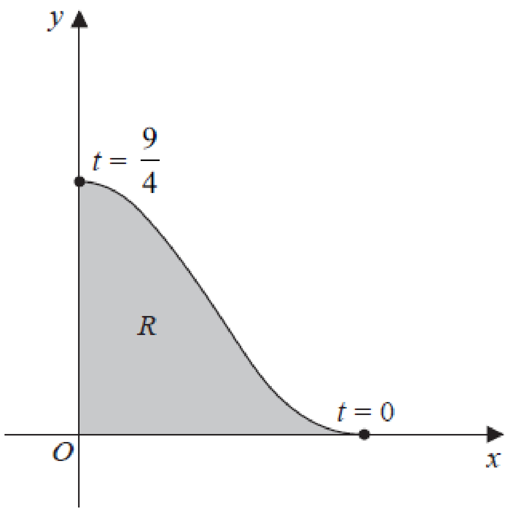


Figure 2

Figure 2 shows a sketch of the curve with parametric equations

$$x = \sqrt{9-4t} \qquad y = \frac{t^3}{\sqrt{9+4t}} \qquad 0 \leqslant t \leqslant \frac{9}{4}$$

The curve touches the x-axis when $t = 0$ and meets the y-axis when $t = \frac{9}{4}$

The region R, shown shaded in Figure 2, is bounded by the curve, the x-axis and the y-axis.

(*a*) Show that the area of R is given by

$$K\int_0^{\frac{9}{4}} \frac{t^3}{\sqrt{81-16t^2}}\,\mathrm{d}t$$

where K is a constant to be found.

(4)

(*b*) Using the substitution $u = 81 - 16t^2$, or otherwise, find the exact area of R.

(*Solutions relying on calculator technology are not acceptable*.)

(6)

(IAL P4, June 2022, Q5)

80.

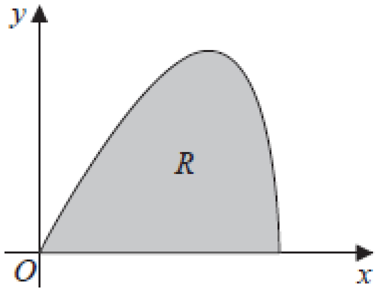


Figure 3

The curve shown in Figure 3 has parametric equations

$$x = 6 \sin t \qquad y = 5 \sin 2t \qquad 0 \leq t \leq \frac{\pi}{2}$$

The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

(*a*) (i) Show that the area of R is given by $\int_0^{\frac{\pi}{2}} 60 \sin t \cos^2 t \, \mathrm{d}t$

(3)

(ii)Hence show, by algebraic integration, that the area of R is exactly 20

(3)

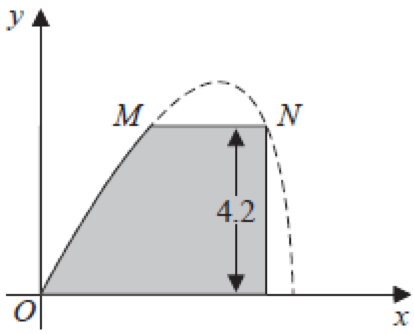


Figure 4

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4.
Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width MN along the top of the dam

(*b*) calculate the width of the walkway.

(5)

(New syllabus P2, Oct 2020, Q12)

81.

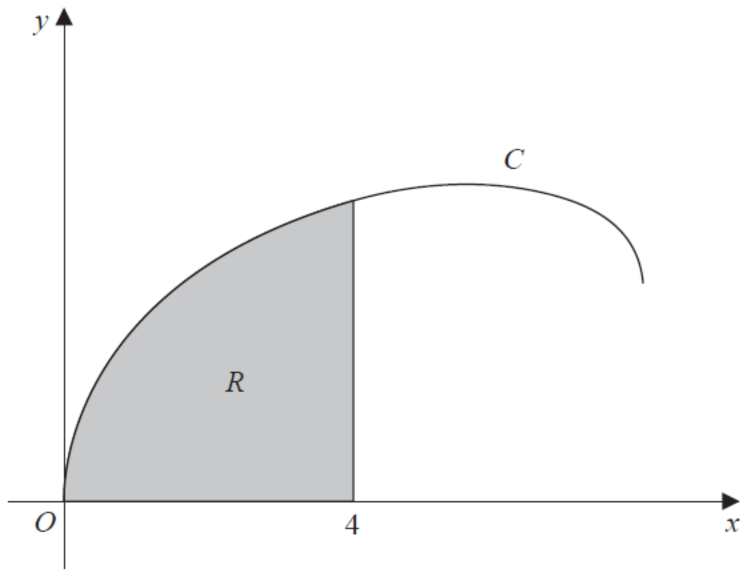


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8\sin^2 t \qquad y = 2\sin 2t + 3\sin t \qquad 0 \leq t \leq \frac{\pi}{2}$$

The region R, shown shaded in Figure 6, is bounded by C, the x-axis and the line with equation $x = 4$

(*a*) Show that the area of R is given by

$$\int_0^a \left(8 - 8\cos 4t + 48\sin^2 t\cos t\right)\mathrm{d}t$$

where a is a constant to be found.

(5)

(*b*) Hence, using algebraic integration, find the exact area of R.

(4)

(New syllabus P1, June 2022, Q16)