OCR Core Maths 3

Past paper questions Integrations

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Integration

- The central idea of calculus is that integration and differentiation are the inverse operations of each other in the same way that plus is the inverse operation of subtraction. In C3 a favourite type of question is to differentiate something using the above rules and then integrate something similar later in the question. View the question as a whole!

- Our basic building blocks for integration are therefore

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c, \qquad \int e^x\,dx = e^x + c, \qquad \int \frac{1}{x}\,dx = \ln x + c.$$

- A big result is gained by inspection below, but worth stating alone:

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c.$$

- Integration by *inspection* is effectively "spotting the answer" by an intermediate guess. The intermediate guess is then differentiated mentally and the final answer should then only be a constant factor out. Things to note are that the power on $e^{\text{something}}$ never changes and be on the lookout for integrals where the top line is *almost* the derivative of the bottom line. Here are a few examples:

Integral	Guess	Answer
$\int (2x+3)^{15}\,dx$	$(2x+3)^{16}+c$	$\frac{1}{32}(2x+3)^{16}+c,$
$\int (1-3x)^{-4}\,dx$	$(1-3x)^{-3}+c$	$\frac{1}{9}(1-3x)^{-3}+c,$
$\int 2\sqrt{4x+1}\,dx$	$(4x+1)^{\frac{3}{2}}+c$	$\frac{1}{3}(4x+1)^{\frac{3}{2}}+c,$
$\int 2e^{3x-5}\,dx$	$e^{3x-5}+c$	$\frac{2}{3}e^{3x-5}+c,$
$\int 7xe^{x^2+1}\,dx$	$e^{x^2+1}+c$	$\frac{7}{2}e^{x^2+1}+c,$
$\int \frac{7}{1-4x}\,dx$	$\ln(1-4x)+c$	$-\frac{7}{4}\ln(1-4x)+c,$
$\int \frac{e^{2x}}{1-e^{2x}}\,dx$	$\ln(1-e^{2x})+c$	$-\frac{1}{2}\ln(1-e^{2x})+c.$

You must practice this a lot... it only comes easily after a while. [Since most students also take C3 and C4 at the same time it is worth noting that all the above can be done by the C4 technique of *integration by substitution*.]

- $\int_a^b \pi y^2\,dx$ is the volume of revolution of the curve y rotated about the x-axis between $x = a$ and $x = b$. All that is needed for you to do is calculate y^2 in terms of x from y. For example find the volume of revolution of the solid formed by rotating the curve $y = \sqrt{2x+3}$ about the x-axis between $x = 10$ and $x = 14$. We need to evaluate $\int_a^b \pi y^2\,dx = \int_{10}^{14} \pi y^2\,dx$. Now the curve is $y = \sqrt{2x+3}$ so to find y^2 in terms of x we need only square the equation $\Rightarrow y^2 = 2x + 3$. We therefore evaluate

$$\int_{10}^{14} \pi y^2\,dx = \pi \int_{10}^{14} (2x+3)\,dx = \pi \left[x^2 + 3x\right]_{10}^{14} = 108\pi.$$

- For volumes of revolution around the y-axis switch the x and the y and use $\int_p^q \pi x^2\,dy$ between $y = p$ and $y = q$. For example find the volume of revolution of the solid formed by rotating the line $y = 3x - 2$ about the y-axis between $y = 0$ and $y = 5$. We need to evaluate $\int_p^q \pi x^2\,dy = \int_0^5 \pi x^2\,dy$. Now the line is $y = 3x - 2$ so to find x^2 in terms of y, we make x the subject and square;

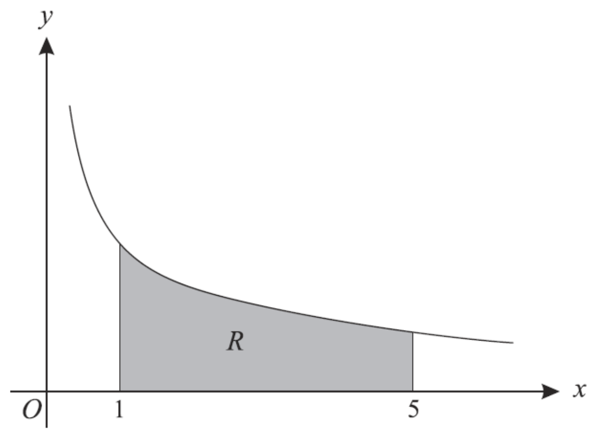
$$y = 3x - 2 \quad \Rightarrow \quad x = \frac{y+2}{3} \quad \Rightarrow \quad x^2 = \frac{y^2 + 4y + 4}{9}.$$

We therefore evaluate

$$\int_0^5 \pi x^2\,dy = \pi \int_0^5 \left(\frac{y^2+4y+4}{9}\right) dy = \frac{\pi}{9}\left[\frac{y^3}{3} + 2y^2 + 4y\right]_0^5 = \frac{335\pi}{27}.$$

1.

(a)



The diagram shows the curve $y = \frac{2}{\sqrt{x}}$. The region R, shaded in the diagram, is bounded by the curve and by the lines $x = 1$, $x = 5$ and $y = 0$. The region R is rotated completely about the x-axis. Find the exact volume of the solid formed. [4]

(b) Use Simpson's rule, with 4 strips, to find an approximate value for

$$\int_1^5 \sqrt{(x^2 + 1)}\,dx,$$

giving your answer correct to 3 decimal places. [4]

Q4 June 2005

2.

Show that $\int_2^8 \frac{3}{x}\,dx = \ln 64.$ [4]

Q1 Jan 2006

3.

Find

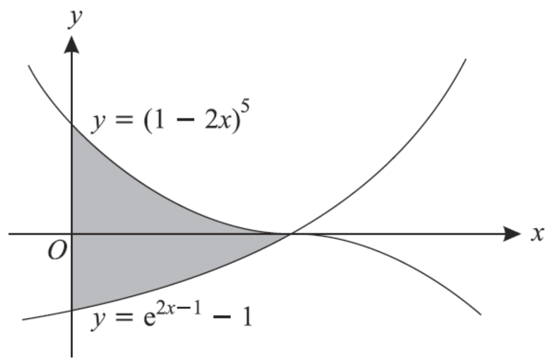
(i) $\int 8e^{-2x}\,dx,$

(ii) $\int (4x + 5)^6\,dx.$

[5]

Q1 Jan 2009

4.



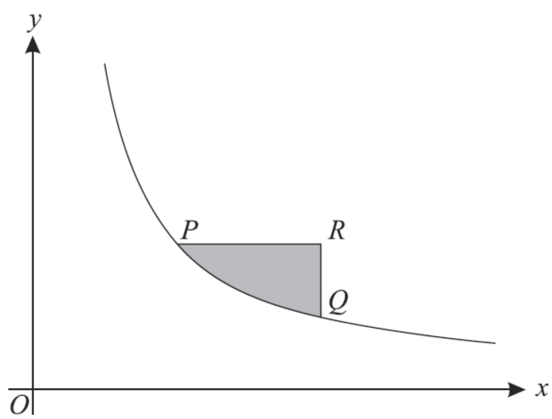
The diagram shows the curves $y = (1 - 2x)^5$ and $y = e^{2x-1} - 1$. The curves meet at the point $(\frac{1}{2}, 0)$. Find the exact area of the region (shaded in the diagram) bounded by the y-axis and by part of each curve. [8]

Q5 Jan 2006

5.

(a) Find the exact value of $\int_1^2 \frac{2}{(4x-1)^2}\,dx$. [4]

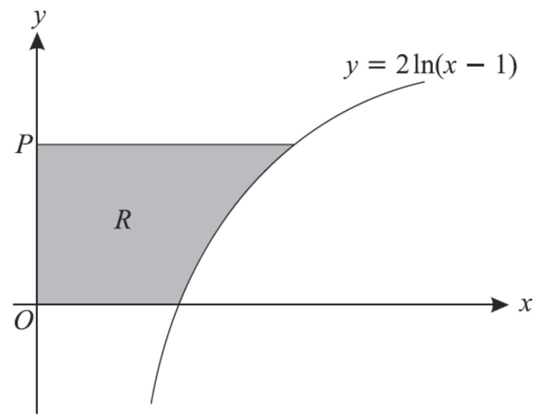
(b)



The diagram shows part of the curve $y = \frac{1}{x}$. The point P has coordinates $\left(a, \frac{1}{a}\right)$ and the point Q has coordinates $\left(2a, \frac{1}{2a}\right)$, where a is a positive constant. The point R is such that PR is parallel to the x-axis and QR is parallel to the y-axis. The region shaded in the diagram is bounded by the curve and by the lines PR and QR. Show that the area of this shaded region is $\ln(\frac{1}{2}e)$. [6]

Q5 June 2006

6.



The diagram shows the curve with equation $y = 2\ln(x-1)$. The point P has coordinates $(0, p)$. The region R, shaded in the diagram, is bounded by the curve and the lines $x = 0$, $y = 0$ and $y = p$. The units on the axes are centimetres. The region R is rotated completely about the **y-axis** to form a solid.

(i) Show that the volume, V cm^3, of the solid is given by

$$V = \pi\left(e^p + 4e^{\frac{1}{2}p} + p - 5\right).$$ [8]

(ii) It is given that the point P is moving in the positive direction along the y-axis at a constant rate of 0.2 cm min^{-1}. Find the rate at which the volume of the solid is increasing at the instant when $p = 4$, giving your answer correct to 2 significant figures. [5]

Q9 June 2006

7.

Find $\displaystyle\int \frac{10}{(2x-7)^2}\,dx$. **[3]**

Q1 Jan 2010

8.

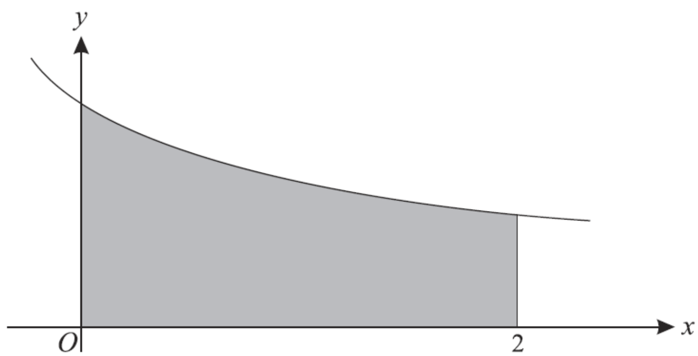
Given that

$$\int_0^{\ln 4} \left(ke^{3x} + (k-2)e^{-\frac{1}{2}x}\right) dx = 185,$$

find the value of the constant k. **[7]**

Q6 Jan 2010

9.



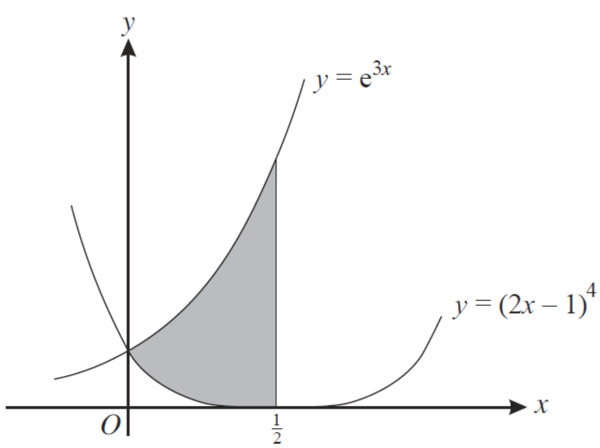
The diagram shows the curve with equation $y = \frac{1}{\sqrt{3x+2}}$. The shaded region is bounded by the curve and the lines $x = 0$, $x = 2$ and $y = 0$.

(i) Find the exact area of the shaded region. [4]

(ii) The shaded region is rotated completely about the x-axis. Find the exact volume of the solid formed, simplifying your answer. [5]

Q6 Jan 2007

10.



The diagram shows the curves $y = e^{3x}$ and $y = (2x - 1)^4$. The shaded region is bounded by the two curves and the line $x = \frac{1}{2}$. The shaded region is rotated completely about the x-axis. Find the exact volume of the solid produced. [9]

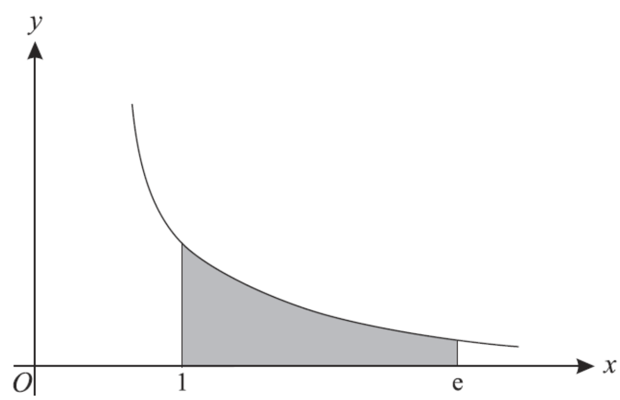
Q6 June 2008

11.

(i) Given that $y = \frac{4\ln x - 3}{4\ln x + 3}$, show that $\frac{dy}{dx} = \frac{24}{x(4\ln x + 3)^2}$. [3]

(ii) Find the exact value of the gradient of the curve $y = \frac{4\ln x - 3}{4\ln x + 3}$ at the point where it crosses the x-axis. [4]

(iii)



The diagram shows part of the curve with equation

$$y = \frac{2}{x^{\frac{1}{2}}(4\ln x + 3)}.$$

The region shaded in the diagram is bounded by the curve and the lines $x = 1$, $x = e$ and $y = 0$. Find the exact volume of the solid produced when this shaded region is rotated completely about the x-axis. [4]

Q8 June 2007

12.

Find

(i) $\int 6e^{2x+1}\, dx$,

(ii) $\int 10(2x+1)^{-1}\, dx$.

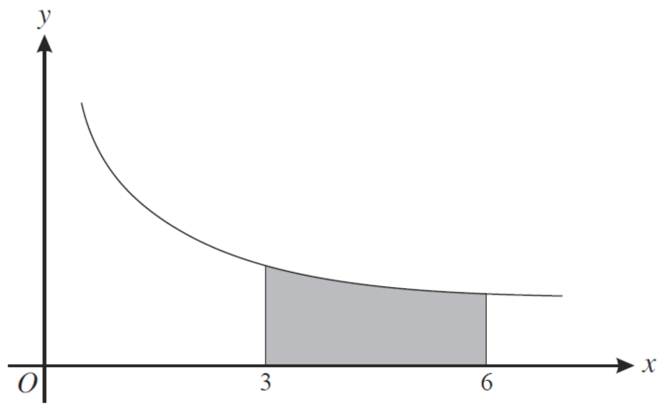
[5]

Q1 June 2011

13.

(a) Find $\int (3x + 7)^9 \, dx$. [3]

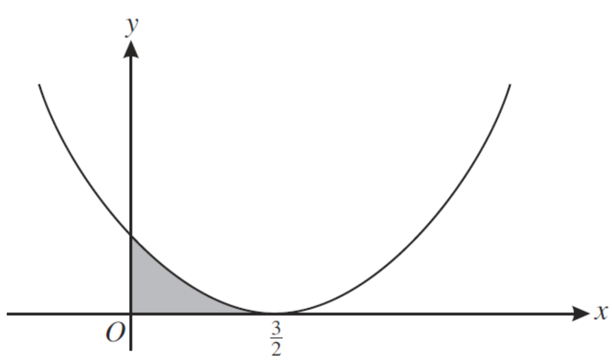
(b)



The diagram shows the curve $y = \frac{1}{2\sqrt{x}}$. The shaded region is bounded by the curve and the lines $x = 3$, $x = 6$ and $y = 0$. The shaded region is rotated completely about the x-axis. Find the exact volume of the solid produced, simplifying your answer. [5]

Q5 Jan 2008

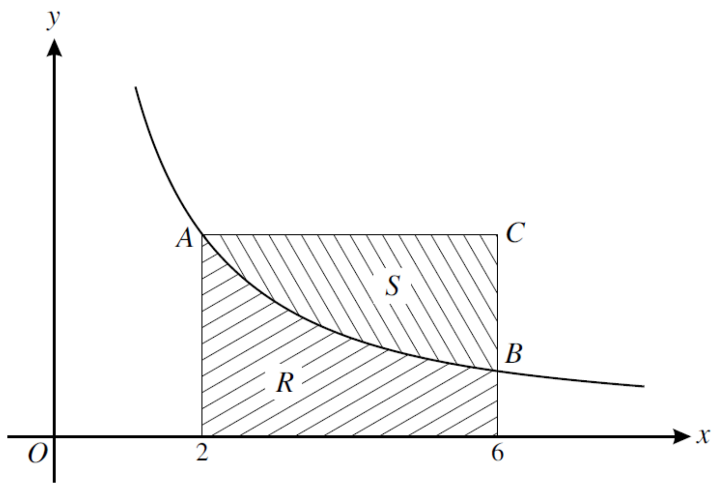
14.



The diagram shows the curve with equation $y = (2x - 3)^2$. The shaded region is bounded by the curve and the lines $x = 0$ and $y = 0$. Find the exact volume obtained when the shaded region is rotated completely about the x-axis. **[5]**

Q2 June 2009

15.



The diagram shows part of the curve $y = \frac{k}{x}$, where k is a positive constant. The points A and B on the curve have x-coordinates 2 and 6 respectively. Lines through A and B parallel to the axes as shown meet at the point C. The region R is bounded by the curve and the lines $x = 2$, $x = 6$ and $y = 0$. The region S is bounded by the curve and the lines AC and BC. It is given that the area of the region R is $\ln 81$.

(i) Show that $k = 4$. **[3]**

(ii) Find the exact volume of the solid produced when the region S is rotated completely about the x-axis. **[4]**

Q4 June 2010

16.

Show that $\int_{\sqrt{2}}^{\sqrt{6}} \frac{2}{x}\,dx = \ln 3$. **[3]**

Q1 Jan 2012

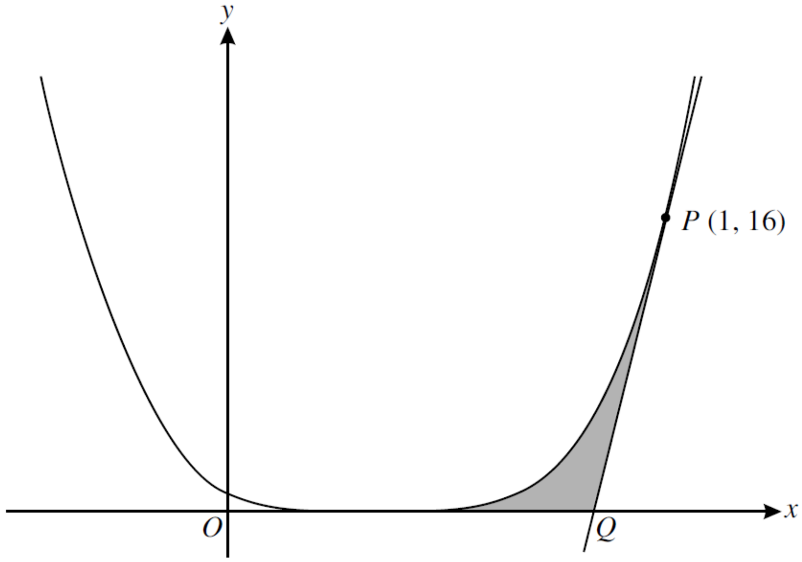
17.

(a) Show that $\int_0^4 \frac{18}{\sqrt{6x+1}}\,dx = 24$. **[4]**

(b) Find $\int_0^1 (e^x + 2)^2\,dx$, giving your answer in terms of e. **[4]**

Q4 June 2012

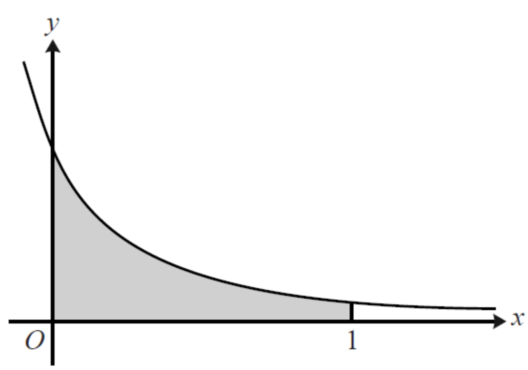
18.



The diagram shows the curve with equation $y = (3x - 1)^4$. The point P on the curve has coordinates $(1, 16)$ and the tangent to the curve at P meets the x-axis at the point Q. The shaded region is bounded by PQ, the x-axis and that part of the curve for which $\frac{1}{3} \leqslant x \leqslant 1$. Find the exact area of this shaded region. **[10]**

Q7 June 2010

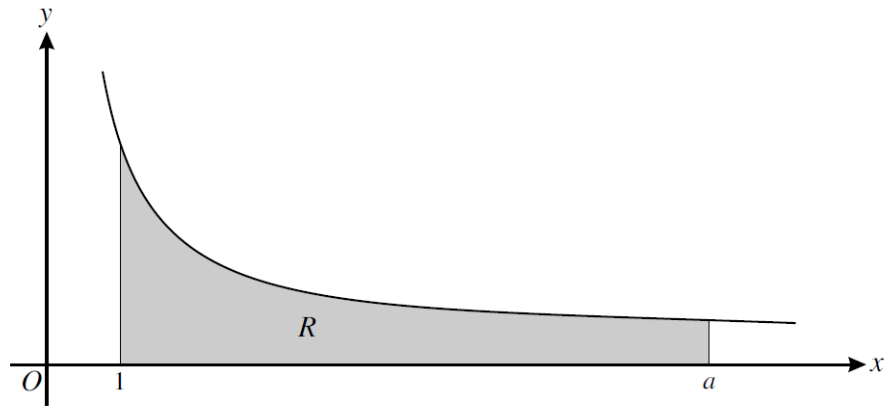
19.



The diagram shows part of the curve $y = \dfrac{6}{(2x+1)^2}$. The shaded region is bounded by the curve and the lines $x = 0$, $x = 1$ and $y = 0$. Find the exact volume of the solid produced when this shaded region is rotated completely about the x-axis. **[5]**

Q2 Jan 2012

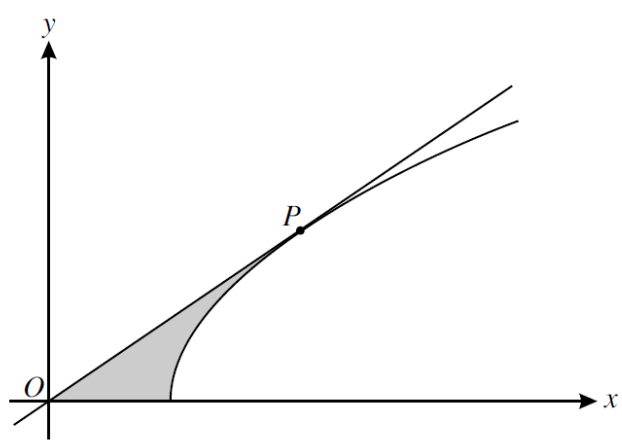
20.



The diagram shows the curve with equation $y = \dfrac{6}{\sqrt{3x-2}}$. The region R, shaded in the diagram, is bounded by the curve and the lines $x = 1$, $x = a$ and $y = 0$, where a is a constant greater than 1. It is given that the area of R is 16 square units. Find the value of a and hence find the exact volume of the solid formed when R is rotated completely about the x-axis. **[9]**

Q5 Jan 2011

21.



The diagram shows the curve with equation $y = \sqrt{3x-5}$. The tangent to the curve at the point P passes through the origin. The shaded region is bounded by the curve, the x-axis and the line OP. Show that the x-coordinate of P is $\frac{10}{3}$ and hence find the exact area of the shaded region. **[9]**

Q6 June 2011

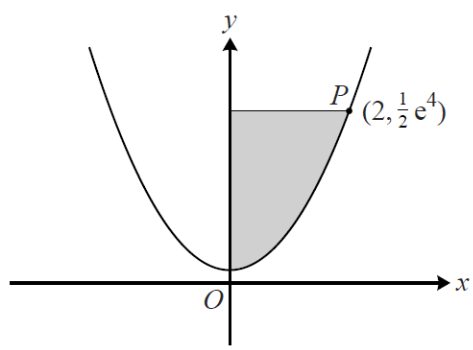
22.

(i) Show that the derivative with respect to y of

$$y\ln(2y) - y$$

is $\ln(2y)$. **[3]**

(ii)

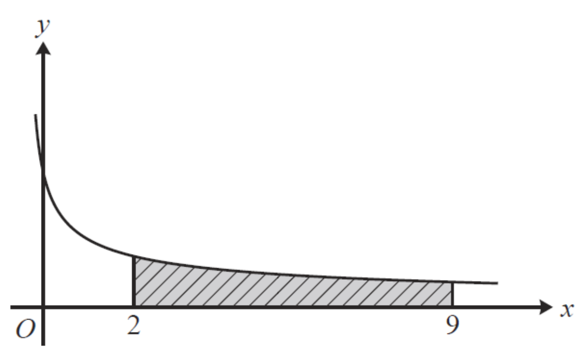


The diagram shows the curve with equation $y = \frac{1}{2}e^{x^2}$. The point $P(2, \frac{1}{2}e^4)$ lies on the curve. The shaded region is bounded by the curve and the lines $x = 0$ and $y = \frac{1}{2}e^4$. Find the exact volume of the solid produced when the shaded region is rotated completely about the y-axis. **[6]**

(iii) Hence find the volume of the solid produced when the region bounded by the curve and the lines $x = 0$, $x = 2$ and $y = 0$ is rotated completely about the y-axis. **[2]**

Q9 June 2012

23.



The diagram shows the curve $y = \dfrac{6}{\sqrt{3x+1}}$. The shaded region is bounded by the curve and the lines $x = 2$, $x = 9$ and $y = 0$.

(i) Show that the area of the shaded region is $4\sqrt{7}$ square units. **[4]**

(ii) The shaded region is rotated completely about the x-axis. Show that the volume of the solid produced can be written in the form $k\ln 2$, where the exact value of the constant k is to be determined. **[5]**

Q5 Jan 2013

24.

Find

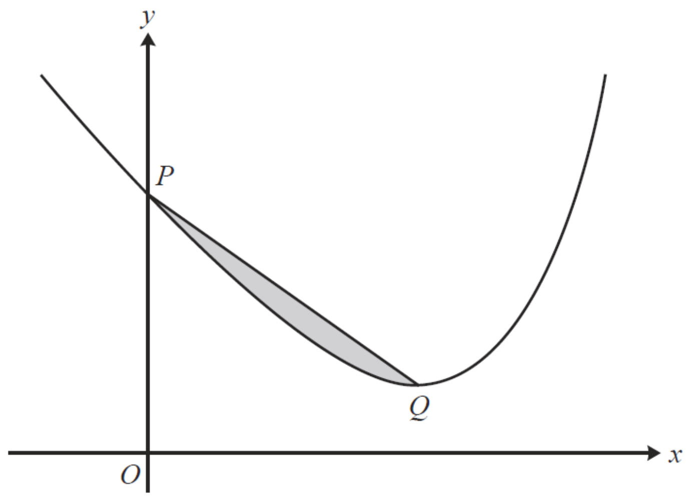
(i) $\int (4 - 3x)^7 \mathrm{d}x$,

(ii) $\int (4 - 3x)^{-1} \mathrm{d}x$.

[5]

Q1 June 2013

25.



The diagram shows the curve

$$y = \mathrm{e}^{2x} - 18x + 15.$$

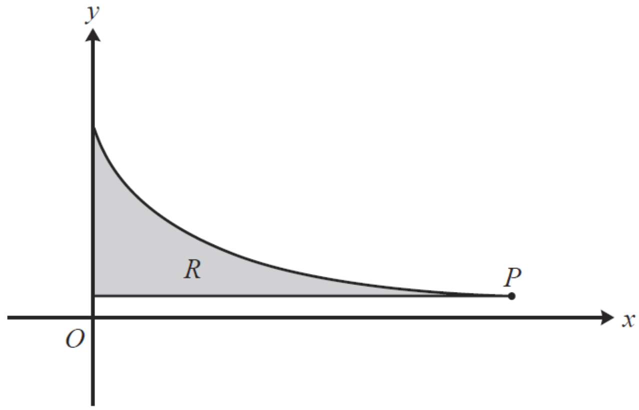
The curve crosses the y-axis at P and the minimum point is Q. The shaded region is bounded by the curve and the line PQ.

(i) Show that the x-coordinate of Q is $\ln 3$. **[3]**

(ii) Find the exact area of the shaded region. **[8]**

Q9 June 2013

26.



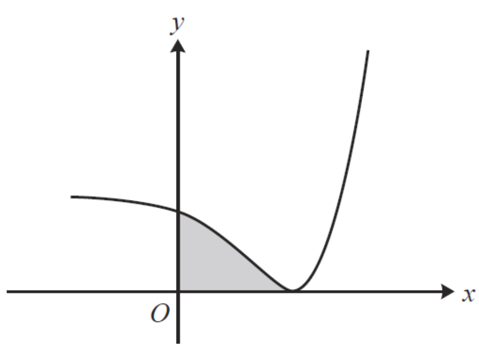
The diagram shows the curve $y = \sqrt{\frac{3}{4x+1}}$ for $0 \leqslant x \leqslant 20$. The point P on the curve has coordinates $\left(20, \frac{1}{9}\sqrt{3}\right)$. The shaded region R is enclosed by the curve and the lines $x = 0$ and $y = \frac{1}{9}\sqrt{3}$.

(i) Find the exact area of R. **[4]**

(ii) Find the exact volume of the solid obtained when R is rotated completely about the x-axis. **[6]**

Q7 June 2014

27.



The diagram shows the curve $y = e^{3x} - 6e^{2x} + 32$.

(i) Find the exact x-coordinate of the minimum point and verify that the y-coordinate of the minimum point is 0. **[4]**

(ii) Find the exact area of the region (shaded in the diagram) enclosed by the curve and the axes. **[4]**

Q5 June 2015