OCR Further Pure 1

Past paper questions
Proof by Induction

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Proof by Induction

- Let $P(n)$ be a proposition which depends on some integer value n. The principle of induction works as follows: Start by demonstrating the truth of $P(1)$ (say). Then we show that *if* $P(k)$ is true for *some value* k *then* it implies the truth of $P(k+1)$, then $P(n)$ must be true for all integer $n \geqslant 1$. This is because we have shown

$$P(1) \Rightarrow P(2), \text{ and } P(2) \Rightarrow P(3), \text{ and } P(3) \Rightarrow P(4) \text{ etc. etc. etc.}$$

- Your answer should always follow this template:
  - "Let $P(n)$ be the proposition that $f(n) = g(n)$ for all $n \geqslant 1$."
  - "Basis Case: If $n = 1$, $f(1) = \dots$ and $g(1) = \dots$. We see $f(1) = g(1)$ so $P(1)$ is true."
  - "Let us suppose that $P(n)$ is true for some $n = k$:

    $$f(k) = g(k)."$$

    [Then manipulate $f(k) = g(k)$ using algebra to obtain the next line]

    $$"f(k+1) = g(k+1)."$$

  - "This is the statement of $P(k+1)$."
  - "Therefore we have shown that *if* $P(k)$ is true *then* $P(k+1)$ is also true and since $P(1)$ is also true we can conclude by the principle of mathematical induction that $P(n)$ is true for all $n \geqslant 1$."

- If an induction question includes a "$\sum$", can I suggest you get rid of it by writing out the sum term-by-term; students tend to get muddled on when to use r, n, k and $k+1$ in my experience (although this might be my teaching). Also leave initial numerical values unevaluated; 1×2^2 is preferable to 4. For example

$$\sum_{r=1}^{n} r(r+2) \Rightarrow 1 \times 3 + 2 \times 4 + \cdots + n(n+2).$$

- For example use induction to prove that for $n \geqslant 2$, $\sum_{r=2}^{n}(r-1)r = \frac{1}{3}n(n-1)(n+1)$.

  - "The question is the same as proving $1\times 2+2\times 3+\cdots+(n-1)n = \frac{1}{3}n(n-1)(n+1)$."
  - "Let $P(n)$ be the proposition $1 \times 2 + 2 \times 3 + \cdots + (n-1)n = \frac{1}{3}n(n-1)(n+1)$."
  - "Basis case: If $n = 2$, $1 \times 2 + 2 \times 3 + \cdots + (n-1)n = 2$ and $\frac{1}{3}n(n-1)(n+1) = 2$. We see that LHS = RHS = 2 so $P(2)$ is true."
  - "Let us suppose that $P(n)$ is true for some $n = k$:

$$1 \times 2 + 2 \times 3 + \cdots + (k-1)k = \frac{1}{3}k(k-1)(k+1)."$$

  - (Add the next term to the LHS to both sides:)

$$\begin{aligned} \text{``}1 \times 2 + 2 \times 3 + \cdots + (k-1)k + \underline{k(k+1)} &= \frac{1}{3}k(k-1)(k+1) + \underline{k(k+1)} \\ &= \frac{1}{3}k(k+1)[(k-1)+3] \\ &= \frac{1}{3}k(k+1)(k+2).\text{''} \end{aligned}$$

  - "This is the statement of $P(k+1)$."
  - "Therefore we have shown that if $P(k)$ is true then $P(k+1)$ is also true and since $P(2)$ is also true we can conclude by the principle of mathematical induction that $P(n)$ is true for all $n \geqslant 2$."

- In a recent official mark scheme, the use of the words 'mathematical induction' in your conclusion was needed for full marks.

1.

Prove by induction that, for $n \geqslant 1$, $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [5]

(Q2, Jan 2006)

2.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(i) Find $\mathbf{A}^2$ and $\mathbf{A}^3$. [3]

(ii) Hence suggest a suitable form for the matrix $\mathbf{A}^n$. [1]

(iii) Use induction to prove that your answer to part **(ii)** is correct. [4]

(Q7, June 2006)

3.

The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_n = n^2 + 3n$, for all positive integers n.

(i) Show that $u_{n+1} - u_n = 2n + 4$. [3]

(ii) Hence prove by induction that each term of the sequence is divisible by 2. [5]

(Q6, Jan 2007)

4.

Prove by induction that, for $n \geqslant 1$, $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$. [5]

(Q2, June 2007)

5.

The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 1$ and $u_{n+1} = u_n + 2n + 1$.

(i) Show that $u_4 = 16$. [2]

(ii) Hence suggest an expression for u_n. [1]

(iii) Use induction to prove that your answer to part **(ii)** is correct. [4]

(Q8, Jan 2008)

6.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geqslant 1$,

$$\mathbf{A}^n = \begin{pmatrix} 3^n & \frac{1}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}.$$ [6]

(Q4, June 2008)

7.

It is given that $u_n = 13^n + 6^{n-1}$, where n is a positive integer.

(i) Show that $u_n + u_{n+1} = 14 \times 13^n + 7 \times 6^{n-1}$. **[3]**

(ii) Prove by induction that u_n is a multiple of 7. **[4]**

(Q7, Jan 2009)

8.

The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 3$ and $u_{n+1} = 3u_n - 2$.

(i) Find u_2 and u_3 and verify that $\frac{1}{2}(u_4 - 1) = 27$. **[3]**

(ii) Hence suggest an expression for u_n. **[2]**

(iii) Use induction to prove that your answer to part **(ii)** is correct. **[5]**

(Q10, June 2009)

9.

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(i) Find $\mathbf{M}^2$ and $\mathbf{M}^3$. **[3]**

(ii) Hence suggest a suitable form for the matrix $\mathbf{M}^n$. **[1]**

(iii) Use induction to prove that your answer to part **(ii)** is correct. **[4]**

(iv) Describe fully the single geometrical transformation represented by $\mathbf{M}^{10}$. **[3]**

(Q10, Jan 2010)

10.

Prove by induction that, for $n \geqslant 1$, $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. **[5]**

(Q1, June 2010)

11.

The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 2$, and $u_{n+1} = 2u_n - 1$ for $n \geqslant 1$. Prove by induction that $u_n = 2^{n-1} + 1$. **[4]**

(Q3, Jan 2011)

12.

Prove by induction that, for $n \geqslant 1$, $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$. **[5]**

(Q2, June 2011)

13.

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$.

(i) Show that $\mathbf{M}^4 = \begin{pmatrix} 81 & 0 \\ 80 & 1 \end{pmatrix}$. **[3]**

(ii) Hence suggest a suitable form for the matrix $\mathbf{M}^n$, where n is a positive integer. **[2]**

(iii) Use induction to prove that your answer to part **(ii)** is correct. **[4]**

(Q7, Jan 2012)

14.

Prove by induction that, for $n \geqslant 1$, $\sum_{r=1}^{n} 4 \times 3^r = 6(3^n - 1)$. **[5]**

(Q5, June 2012)

15.

The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 2$ and $u_{n+1} = \dfrac{u_n}{1 + u_n}$ for $n \geqslant 1$.

(i) Find u_2 and u_3, and show that $u_4 = \frac{2}{7}$. **[3]**

(ii) Hence suggest an expression for u_n. **[2]**

(iii) Use induction to prove that your answer to part **(ii)** is correct. **[5]**

(Q10, Jan 2013)

16.

The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geqslant 1$,

$$\mathbf{M}^n = \begin{pmatrix} 2^n & 2^{n+1} - 2 \\ 0 & 1 \end{pmatrix}.$$ **[6]**

(Q4, June 2013)

17.

The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_n = 5^n + 2^{n-1}$.

(i) Find u_1, u_2 and u_3. **[2]**

(ii) Hence suggest a positive integer, other than 1, which divides exactly into every term of the sequence. **[1]**

(iii) By considering $u_{n+1} + u_n$, prove by induction that your suggestion in part **(ii)** is correct. **[5]**

(Q10, June 2014)

18.

Prove by induction that, for $n \geqslant 1$, $\displaystyle\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$. **[5]**

(Q4, June 2015)