OCR Core Maths 3

Past paper questions Functions

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Functions

- A *function* is a *one-to-one* or a *many-to-one mapping*. There are also *many-to-many* and *one-to-many* mappings, but these are ***not*** functions. In a function, for every value you feed into the function you obtain one (and only one) value out.

- The *domain* of a function $y = f(x)$ is all the possible values of x the function can take. For example the domain of $y = \sqrt{x-4}$ is $x \geqslant 4$. In other words all the *inputs* the function can take.

- The *range* of a function is all the possible *outputs*. That is all the possible values of $f(x)$. So for $f(x) = -x^2 + 5$ the range is $f(x) \leqslant 5$.

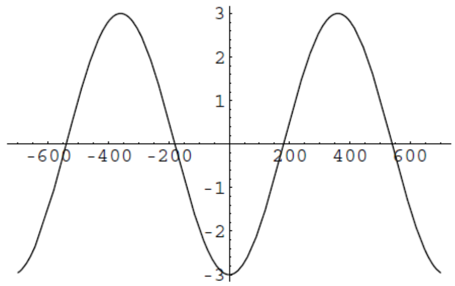
- Functions are transformed as follows

FUNCTION	GRAPH SHAPE
$f(x)$	Normal Graph
$2f(x)$	Graph stretched by a factor of 2 parallel to the y-axis i.e. every value of $f(x)$ in the original graph is multiplied by 2
$f(2x)$	Graph stretched by factor of $\frac{1}{2}$ parallel to the x-axis
$3f(\frac{x}{4})$	Graph stretched by factor of 4 parallel to the x-axis and a stretch by a factor of 3 parallel to the y-axis
$f(x)+6$	Graph translated vertically *up* 6 units
$f(x)-6$	Graph translated vertically *down* 6 units
$f(x+4)$	Graph translated 4 units to the *left*
$f(x-6)$	Graph translated 6 units to the *right*
$f(x-6)+9$	Graph translated 6 units to the *right* and 9 units *up*. This is a translation and can be expressed as $\binom{6}{9}$ where $\binom{\text{change in } x}{\text{change in } y}$
$-f(x)$	Graph reflected in the x-axis
$f(-x)$	Graph reflected in the y-axis

- When faced with more than one of the above transformations it sometimes matters which order you carry out the transformations. In the example of $2f(x-3)$ it doesn't matter because you end up with the same result both ways, regardless of whether you do the translation right, or the stretch parallel to the y-axis first (think about it). However with $f(2x+10)$ you get a different result depending on the order you carry out the translation 10 left and then stretch factor $\frac{1}{2}$ parallel to the x-axis. If the conflict occurs within the bracket you should do the *opposite* of what you expect. So here you do the translation first and then the stretch.

  For $2f(x)+6$ the transformations are outside the bracket, so here you would do the stretch *then* the translation.

- So for example if you were asked to sketch $y = 3\sin(\frac{x}{2} - 90)$ you would translate '$y = \sin x$' $90°$ to the right, *then* stretch factor 2 parallel to the x-axis and stretch factor 3 parallel to the y-axis.



- If $f(x) = f(-x)$ then the function is called an *even* function. An even function is one where the y-axis is a line of symmetry. Examples are

$$f(x) = \cos x \qquad \text{since} \qquad f(-x) = \cos(-x) = \cos x = f(x),$$
$$g(x) = x^2 + 1 \qquad \text{since} \qquad g(-x) = (-x)^2 + 1 = x^2 + 1 = g(x).$$

- If $-f(x) = f(-x)$ then the function is called an *odd* function. An odd function is one where the function is unchanged if you rotate it $180°$ around the point $(0, 0)$. Examples are

$$f(x) = \sin x \qquad \text{since} \qquad f(-x) = \sin(-x) = -\sin x = -f(x),$$
$$g(x) = x^3 \qquad \text{since} \qquad g(-x) = (-x)^3 = -x^3 = -g(x).$$

- You must be able to construct compositions of functions. Note that $f(g(x))$ is not usually the same as $g(f(x))$. For example if $f(x) = x^2$ and $g(x) = x+1$ then $f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$. Contrast this with $g(f(x)) = g(x^2) = x^2 + 1$.

- Sometimes you will be asked to describe a quadratic of the form $ax^2 + bx + c$ in terms of $f(x) = x^2$. It is often useful to *complete the square*. Very quickly I will go through a couple of examples of how to do this:

$x^2 + 10 \Rightarrow$ Clearly just $f(x) + 10$.

$x^2 + 6x + 10 \Rightarrow$ Complete square to get $(x+3)^2 - 9 + 10 = (x+3)^2 + 1$ so it is $f(x+3) + 1$, which is the translation $\binom{-3}{1}$ of x^2.

$2x^2 + 16x + 1 \Rightarrow$ Complete square to get $2(x+4)^2 - 31$ so it is $2f(x+4) - 31$, which is a stretch of factor 2 away from the x-axis, followed by a translation $\binom{-4}{-31}$ of x^2.

- The inverse of a function $f(x)$ is denoted $f^{-1}(x)$. To find the inverse of a function you swap round the x and the y and make y the subject again. This will be the inverse of the original function. For example find the inverse of $f(x) = \sqrt{x^3 + 2}$ gives

$$\begin{aligned} f(x) &= \sqrt{x^3+2}, \\ \Rightarrow \qquad y &= \sqrt{x^3+2}, \\ \Rightarrow \qquad x &= \sqrt{y^3+2}, \\ \Rightarrow \qquad y &= \sqrt[3]{x^2-2}, \\ \Rightarrow \qquad f^{-1}(x) &= \sqrt[3]{x^2-2}. \end{aligned}$$

- A function only has an inverse if it is a one-to-one mapping. If the original function is a many-to-one function (e.g. $y = x^2$ or any of the trig functions) you must restrict its domain to make it a one-to-one mapping (e.g. for $y = x^2$ restrict domain to $x \geqslant 0$). The domain and range of a function are switched in its inverse. For example if $f(x)$ has domain $x > 8$ and range $f(x) \leqslant -10$, then its inverse $f^{-1}(x)$ has domain $x \leqslant -10$ and range $f^{-1}(x) > 8$.

- Geometrically the relationship between a function and its inverse is a reflection in the line $y = x$. A useful spin-off from this result is that if you are asked to find where a function equals its inverse (i.e. $f(x) = f^{-1}(x)$) all you need to do is solve $f(x) = x$ or $f^{-1}(x) = x$; take your pick.

- Given a point on a function ($(3, 4)$, say) then the equivalent point on its inverse is $(4, 3)$ because it has been reflected in $y = x$. If the gradient at $(3, 4)$ was 7, then the gradient on the inverse will be its reciprocal $\frac{1}{7}$.

Modulus

- The modulus function makes everything you put into it positive. For example $|4| = 4$ and $|-6| = 6$. If something negative is 'fed in' to the mod function then it multiplies it by -1 to turn it positive; otherwise it leaves it alone.

- If you have an expression such as $|x-4|$, then the critical value for x is $x = 4$; if $x > 4$ then the expression is just $x-4$ and if $x < 4$ then the expression becomes $-x+4$ because the mod function multiplies it by -1 to turn it positive. This idea helps us solve modulus equations; for example to solve $|2x-1| = 6$ we first look for the critical values of x; here clearly $x = \frac{1}{2}$. We therefore set up two equations depending on whether $x > \frac{1}{2}$ or $x < \frac{1}{2}$:

$$\begin{aligned} \text{If } x &< \tfrac{1}{2} \\ \text{then } -2x+1 &= 6 \\ x &= -\tfrac{5}{2}, \end{aligned} \qquad \begin{aligned} \text{If } x &> \tfrac{1}{2} \\ \text{then } 2x-1 &= 6 \\ x &= \tfrac{7}{2}. \end{aligned}$$

  We perform a little check at the end to check that the solutions found actually satisfy the conditions on x are met; the left hand equation is valid if $x < \frac{1}{2}$ and the solution we have found *is* less that $\frac{1}{2}$; the right hand equation is valid if $x > \frac{1}{2}$ and the solution *is* greater than $\frac{1}{2}$. Therefore both solutions found are valid.

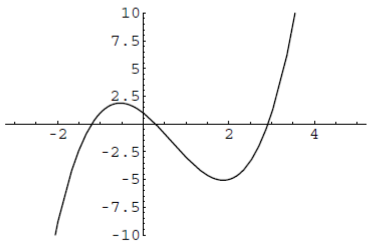
- If you merely have an equation such as $|\text{something}| = |\text{something else}|$ then just get rid of the mods and square both sides to get $(\text{something})^2 = (\text{something else})^2$. *Check* your answers back in the original mod equation to check they work!

- Consider the intimidating looking $|2x-1| - 1 < |x+2|$. As with most inequalities a good first step is to solve the *equality*; i.e. solve $|2x-1| - 1 = |x+2|$. The critical x values are $x = -2$ and $x = \frac{1}{2}$ so we need to set up three different equations depending whether x is $x < -2$, $-2 < x < \frac{1}{2}$ or $x > \frac{1}{2}$ and solve:

$$\begin{aligned} \text{If } x &< -2 \\ \text{then } -2x+1-1 &= -x-2 \\ x &= 2, \end{aligned} \qquad \begin{aligned} \text{If } -2 < x &< \tfrac{1}{2} \\ \text{then } -2x+1-1 &= x+2 \\ x &= -\tfrac{2}{3}, \end{aligned} \qquad \begin{aligned} \text{If } x &> \tfrac{1}{2} \\ \text{then } 2x-1-1 &= x+2 \\ x &= 4. \end{aligned}$$

  Performing our check again we see that two solutions are fine, but $x = 2$ is *not* a solution because the equation was only valid if $x < -2$. Therefore the solution of the equation is $x = -\frac{2}{3}$ or $x = 4$. To solve the inequality we need to see if a number less than $-\frac{2}{3}$ works in the inequality (it doesn't), to see if a number between $-\frac{2}{3}$ and 4 works (it does) and to see if a number greater than 4 works (it doesn't). Therefore $-\frac{2}{3} < x < 4$ is the solution to the question.

- Given a graph of $y = f(x)$ you must be able to draw the graph of $y = |f(x)|$; this is done by leaving any parts of the curve above the x-axis where they are and reflecting parts of the curve under the x-axis so that the are above the x-axis. In the reflected parts, the equation of the curve would be $y = -f(x)$. For example:



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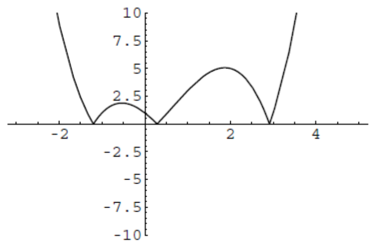

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1.

The function f is defined for all real values of x by

$$f(x) = 10 - (x + 3)^2.$$

(i) State the range of f. [1]

(ii) Find the value of $ff(-1)$. [3]

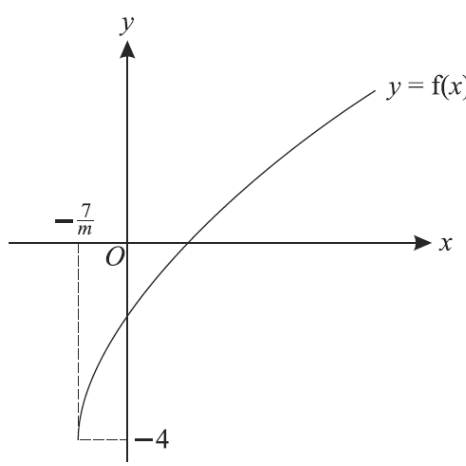
Q1 June 2005

2.

Find the exact solutions of the equation $|6x - 1| = |x - 1|$. [4]

Q2 June 2005

3.



The function f is defined by $f(x) = \sqrt{(mx + 7)} - 4$, where $x \geqslant -\frac{7}{m}$ and m is a positive constant. The diagram shows the curve $y = f(x)$.

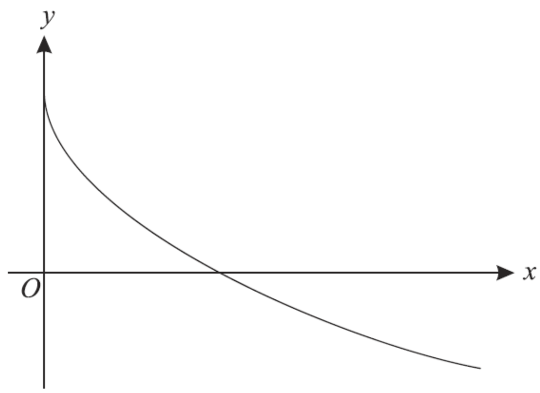
(i) A sequence of transformations maps the curve $y = \sqrt{x}$ to the curve $y = f(x)$. Give details of these transformations. [4]

(ii) Explain how you can tell that f is a one-one function and find an expression for $f^{-1}(x)$. [4]

(iii) It is given that the curves $y = f(x)$ and $y = f^{-1}(x)$ do not meet. Explain how it can be deduced that neither curve meets the line $y = x$, and hence determine the set of possible values of m. [5]

Q9 June 2005

4.



The function f is defined by $f(x) = 2 - \sqrt{x}$ for $x \geqslant 0$. The graph of $y = f(x)$ is shown above.

(i) State the range of f. [1]

(ii) Find the value of $ff(4)$. [2]

(iii) Given that the equation $|f(x)| = k$ has two distinct roots, determine the possible values of the constant k. [2]

Q4 Jan 2006

5.

Solve the inequality $|2x - 3| < |x + 1|$. [5]

Q2 June 2006

6.

The curve $y = \ln x$ is transformed to the curve $y = \ln\left(\frac{1}{2}x - a\right)$ by means of a translation followed by a stretch. It is given that a is a positive constant.

(i) Give full details of the translation and stretch involved. [2]

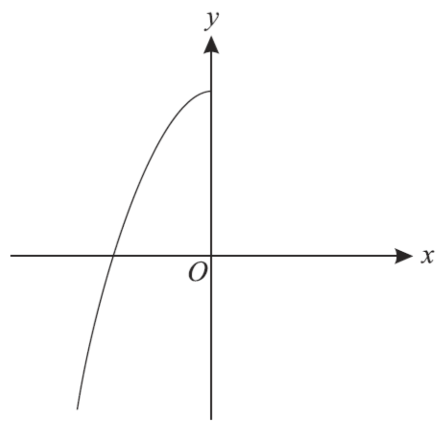
(ii) Sketch the graph of $y = \ln\left(\frac{1}{2}x - a\right)$. [2]

(iii) Sketch, on another diagram, the graph of $y = \left|\ln\left(\frac{1}{2}x - a\right)\right|$. [2]

(iv) State, in terms of a, the set of values of x for which $\left|\ln\left(\frac{1}{2}x - a\right)\right| = -\ln\left(\frac{1}{2}x - a\right)$. [2]

Q7 Jan 2007

7.



The diagram shows the graph of $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2 - x^2, \qquad x \leqslant 0.$$

(i) Evaluate $\mathrm{ff}(-3)$. [3]

(ii) Find an expression for $\mathrm{f}^{-1}(x)$. [3]

(iii) Sketch the graph of $y = \mathrm{f}^{-1}(x)$. Indicate the coordinates of the points where the graph meets the axes. [3]

Q6 June 2006

8.

Functions f and g are defined by

$$\mathrm{f}(x) = 2\sin x \quad \text{for } -\tfrac{1}{2}\pi \leqslant x \leqslant \tfrac{1}{2}\pi,$$
$$\mathrm{g}(x) = 4 - 2x^2 \quad \text{for } x \in \mathbb{R}.$$

(i) State the range of f and the range of g. [2]

(ii) Show that $\mathrm{gf}(0.5) = 2.16$, correct to 3 significant figures, and explain why $\mathrm{fg}(0.5)$ is not defined. [4]

(iii) Find the set of values of x for which $\mathrm{f}^{-1}\mathrm{g}(x)$ is not defined. [6]

Q9 Jan 2007

9.

Solve the inequality $|4x - 3| < |2x + 1|$. [5]

Q2 June 2007

10.

The function f is defined for all non-negative values of x by

$$f(x) = 3 + \sqrt{x}.$$

(i) Evaluate ff(169). [2]

(ii) Find an expression for $f^{-1}(x)$ in terms of x. [2]

(iii) On a single diagram sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, indicating how the two graphs are related. [3]

Q3 June 2007

11.

Functions f and g are defined for all real values of x by

$$f(x) = x^3 + 4 \quad \text{and} \quad g(x) = 2x - 5.$$

Evaluate

(i) fg(1), [2]

(ii) $f^{-1}(12)$. [3]

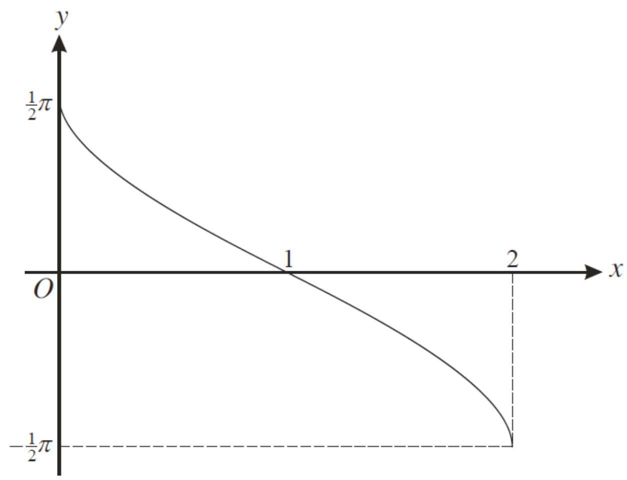
Q1 Jan 2008

12.

Find the exact solutions of the equation $|4x - 5| = |3x - 5|$. [4]

Q1 June 2008

13.



The diagram shows the graph of $y = -\sin^{-1}(x-1)$.

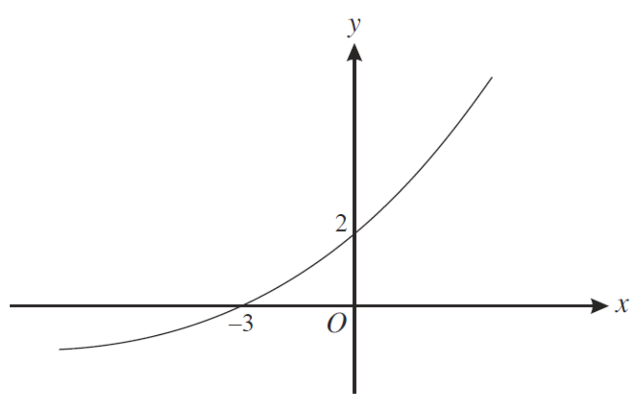
(i) Give details of the pair of geometrical transformations which transforms the graph of $y = -\sin^{-1}(x-1)$ to the graph of $y = \sin^{-1} x$. [3]

(ii) Sketch the graph of $y = |-\sin^{-1}(x-1)|$. [2]

(iii) Find the exact solutions of the equation $|-\sin^{-1}(x-1)| = \frac{1}{3}\pi$. [3]

Q6 Jan 2008

14.



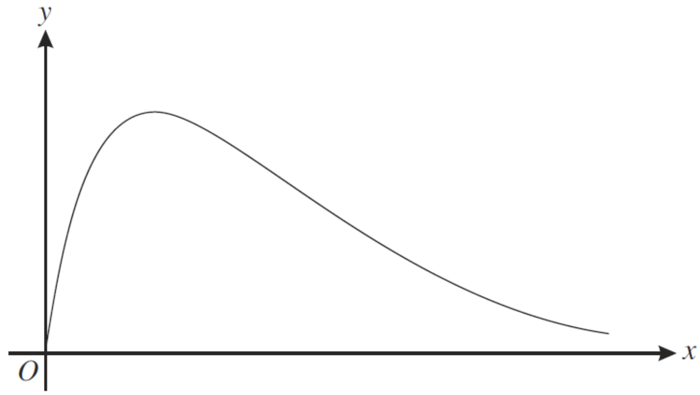
The diagram shows the graph of $y = f(x)$. It is given that $f(-3) = 0$ and $f(0) = 2$. Sketch, on separate diagrams, the following graphs, indicating in each case the coordinates of the points where the graph crosses the axes:

(i) $y = f^{-1}(x)$, [2]

(ii) $y = -2f(x)$. [3]

Q2 June 2008

15.



The function f is defined for the domain $x \geqslant 0$ by

$$f(x) = \frac{15x}{x^2 + 5}.$$

The diagram shows the curve with equation $y = f(x)$.

(i) Find the range of f. [6]

(ii) The function g is defined for the domain $x \geqslant k$ by

$$g(x) = \frac{15x}{x^2 + 5}.$$

Given that g is a one-one function, state the least possible value of k. [1]

(iii) Show that there is no point on the curve $y = g(x)$ at which the gradient is -1. [4]

Q9 June 2008

16.

The functions f and g are defined for all real values of x by

$$f(x) = 3x - 2 \quad \text{and} \quad g(x) = 3x + 7.$$

Find the exact coordinates of the point at which

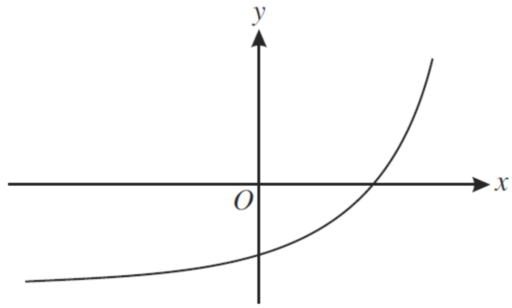
(i) the graph of $y = fg(x)$ meets the x-axis, **[3]**

(ii) the graph of $y = g(x)$ meets the graph of $y = g^{-1}(x)$, **[3]**

(iii) the graph of $y = |f(x)|$ meets the graph of $y = |g(x)|$. **[4]**

Q5 June 2009

17.



The diagram shows the curve $y = e^{kx} - a$, where k and a are constants.

(i) Give details of the pair of transformations which transforms the curve $y = e^x$ to the curve $y = e^{kx} - a$. **[3]**

(ii) Sketch the curve $y = |e^{kx} - a|$. **[2]**

(iii) Given that the curve $y = |e^{kx} - a|$ passes through the points $(0, 13)$ and $(\ln 3, 13)$, find the values of k and a. **[4]**

Q7 Jan 2009

18.

The transformations R, S and T are defined as follows.

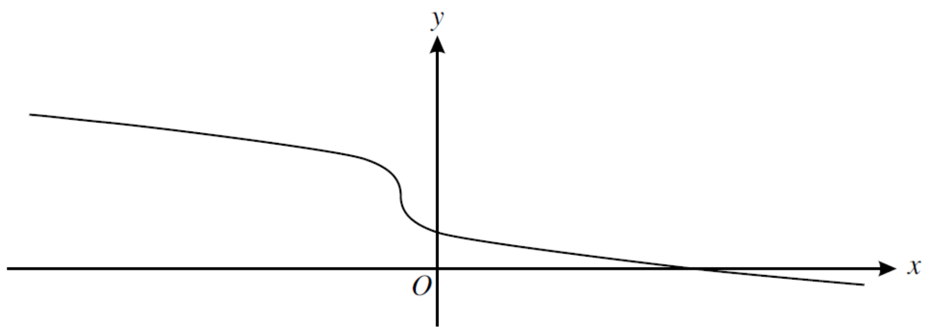
R : reflection in the x-axis
S : stretch in the x-direction with scale factor 3
T : translation in the positive x-direction by 4 units

(i) The curve $y = \ln x$ is transformed by R followed by T. Find the equation of the resulting curve. **[2]**

(ii) Find, in terms of S and T, a sequence of transformations that transforms the curve $y = x^3$ to the curve $y = \left(\frac{1}{9}x - 4\right)^3$. You should make clear the order of the transformations. **[2]**

Q2 June 2010

18.



The function f is defined for all real values of x by

$$f(x) = 2 - \sqrt[3]{x+1}.$$

The diagram shows the graph of $y = f(x)$.

(i) Evaluate $ff(-126)$. **[2]**

(ii) Find the set of values of x for which $f(x) = |f(x)|$. **[2]**

(iii) Find an expression for $f^{-1}(x)$. **[3]**

(iv) State how the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are related geometrically. **[1]**

Q4 Jan 2010

19.

(i) Solve the inequality $|2x + 1| \leqslant |x - 3|$. **[5]**

(ii) Given that x satisfies the inequality $|2x + 1| \leqslant |x - 3|$, find the greatest possible value of $|x + 2|$. **[2]**

Q4 June 2010

20.

Solve the equation $|3x + 4a| = 5a$, where a is a positive constant. **[3]**

Q1 Jan 2011

21.

Solve the inequality $|2x - 5| > |x + 1|$. **[5]**

Q1 June 2012

22.

The functions f and g are defined for all real values of x by

$$f(x) = 4x^2 - 12x \qquad \text{and} \qquad g(x) = ax + b,$$

where a and b are non-zero constants.

(i) Find the range of f. **[3]**

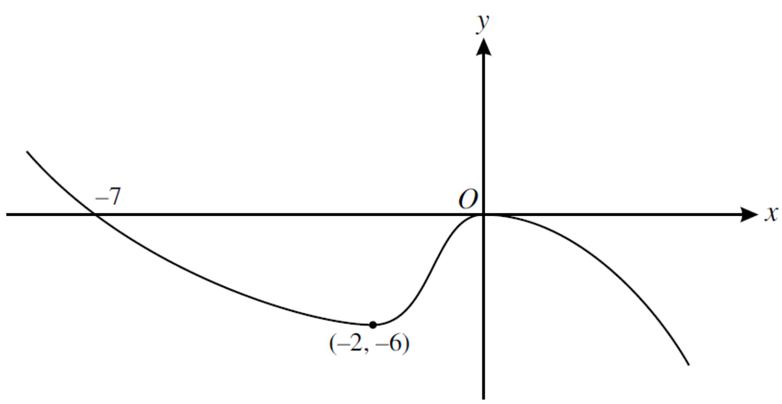
(ii) Explain why the function f has no inverse. **[2]**

(iii) Given that $g^{-1}(x) = g(x)$ for all values of x, show that $a = -1$. **[4]**

(iv) Given further that $gf(x) < 5$ for all values of x, find the set of possible values of b. **[4]**

Q9 June 2010

23.



The diagram shows the curve with equation $y = f(x)$. It is given that $f(-7) = 0$ and that there are stationary points at $(-2, -6)$ and $(0, 0)$. Sketch the curve with equation $y = -4f(x + 3)$, indicating the coordinates of the stationary points. **[4]**

Q2 Jan 2011

24.

The curve $y = \ln x$ is transformed by:

a reflection in the x-axis,
followed by a stretch with scale factor 3 parallel to the y-axis,
followed by a translation in the positive y-direction by $\ln 4$.

Find the equation of the resulting curve, giving your answer in the form $y = \ln(f(x))$. **[4]**

Q2 June 2011

25.

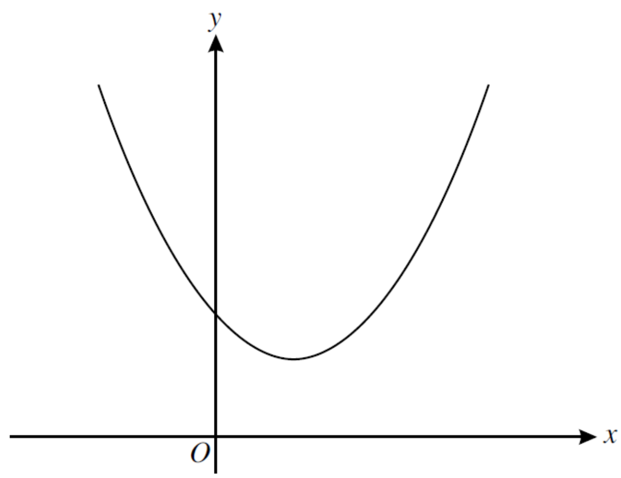
(i) The function f is defined for all real values of x by

$$f(x) = e^{2x} - 3e^{-2x}.$$

(a) Show that $f'(x) > 0$ for all x. **[3]**

(b) Show that the set of values of x for which $f''(x) > 0$ is the same as the set of values of x for which $f(x) > 0$, and state what this set of values is. **[5]**

(ii)



The function g is defined for all real values of x by

$$g(x) = e^{2x} + ke^{-2x},$$

where k is a constant greater than 1. The graph of $y = g(x)$ is shown above. Find the range of g, giving your answer in simplified form. **[5]**

Q9 Jan 2011

26.

The functions f, g and h are defined for all real values of x by

$$f(x) = |x|, \qquad g(x) = 3x + 5 \qquad \text{and} \qquad h(x) = gg(x).$$

(i) Solve the equation $g(x + 2) = f(-12)$. **[3]**

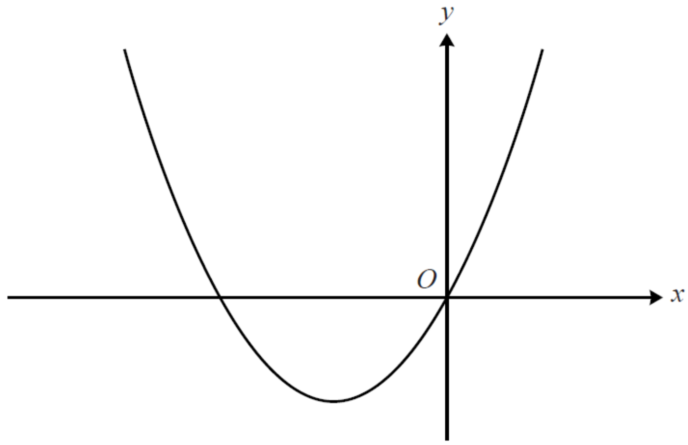
(ii) Find $h^{-1}(x)$. **[3]**

(iii) Determine the values of x for which

$$x + f(x) = 0.$$ **[2]**

Q7 June 2011

27.



The function f is defined for all real values of x by

$$f(x) = k(x^2 + 4x),$$

where k is a positive constant. The diagram shows the curve with equation $y = f(x)$.

(i) The curve $y = x^2$ can be transformed to the curve $y = f(x)$ by the following sequence of transformations:

a translation parallel to the x-axis,
a translation parallel to the y-axis,
a stretch.

Give details, in terms of k where appropriate, of these transformations. **[5]**

(ii) Find the range of f in terms of k. **[2]**

(iii) It is given that there are three distinct values of x which satisfy the equation $|f(x)| = 20$. Find the value of k and determine exactly the three values of x which satisfy the equation in this case. **[6]**

Q9 Jan 2012

28.

The function f is defined for all real values of x by $f(x) = 2x + 5$. The function g is defined for all real values of x and is such that $g^{-1}(x) = \sqrt[3]{x - a}$, where a is a constant. It is given that $fg^{-1}(12) = 9$. Find the value of a and hence solve the equation $gf(x) = 68$. **[7]**

Q7 June 2012

29.

(a) Given that $|t| = 3$, find the possible values of $|2t - 1|$. **[3]**

(b) Solve the inequality $|x - \sqrt{2}| > |x + 3\sqrt{2}|$. **[4]**

Q3 Jan 2013

30.

The functions f and g are defined for all real values of x by

$$f(x) = x^2 + 4ax + a^2 \quad \text{and} \quad g(x) = 4x - 2a,$$

where a is a positive constant.

(i) Find the range of f in terms of a. **[4]**

(ii) Given that $fg(3) = 69$, find the value of a and hence find the value of x such that $g^{-1}(x) = x$. **[6]**

Q8 Jan 2013

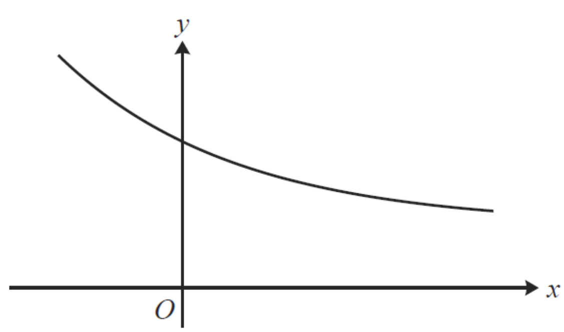
31.

(i) Give full details of a sequence of two transformations needed to transform the graph of $y = |x|$ to the graph of $y = |2(x+3)|$. **[3]**

(ii) Solve the inequality $|x| > |2(x+3)|$, showing all your working. **[5]**

Q5 June 2013

32.



The diagram shows the curve $y = f(x)$, where f is the function defined for all real values of x by

$$f(x) = 3 + 4e^{-x}.$$

(i) State the range of f. **[1]**

(ii) Find an expression for $f^{-1}(x)$, and state the domain and range of f^{-1}. **[4]**

(iii) The straight line $y = x$ meets the curve $y = f(x)$ at the point P. By using an iterative process based on the equation $x = f(x)$, with a starting value of 3, find the coordinates of the point P. Show all your working and give each coordinate correct to 3 decimal places. **[4]**

(iv) How is the point P related to the curves $y = f(x)$ and $y = f^{-1}(x)$? **[1]**

Q7 June 2013

33.

The functions f and g are defined for all real values of x by

$$f(x) = 2x^3 + 4 \quad \text{and} \quad g(x) = \sqrt[3]{x - 10}\ .$$

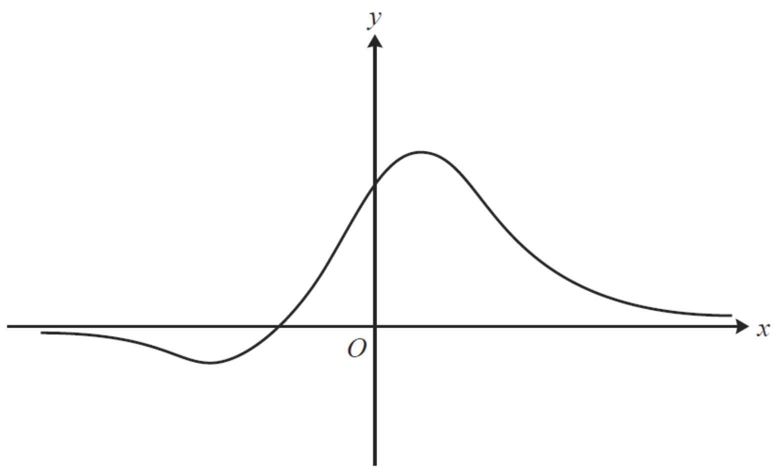
(i) Evaluate $f^{-1}(-50)$. **[2]**

(ii) Show that $fg(x) = 2x - 16$. **[2]**

(iii) Differentiate $gf(x)$ with respect to x. **[3]**

Q4 June 2014

34.



The diagram shows the curve $y = \dfrac{2x+4}{x^2+5}$.

(i) Find $\dfrac{dy}{dx}$ and hence find the coordinates of the two stationary points. **[6]**

(ii) The function g is defined for all real values of x by

$$g(x) = \left|\frac{2x+4}{x^2+5}\right| .$$

(a) Sketch the curve $y = g(x)$ and state the range of g. **[3]**

(b) It is given that the equation $g(x) = k$, where k is a constant, has exactly two distinct real roots. Write down the set of possible values of k. **[2]**

Q8 June 2014

35.

It is given that $|x+3a| = 5a$, where a is a positive constant. Find, in terms of a, the possible values of

$$|x+7a| - |x-7a|.$$ **[6]**

Q4 June 2015

36.

The functions f and g are defined as follows:

$$f(x) = 2 + \ln(x+3) \text{ for } x \geqslant 0,$$

$$g(x) = ax^2 \text{ for all real values of } x \text{, where } a \text{ is a positive constant.}$$

(i) Given that $gf(e^4 - 3) = 9$, find the value of a. **[3]**

(ii) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. **[3]**

(iii) Given that $ff(e^N - 3) = \ln(53e^2)$, find the value of N. **[5]**

Q8 June 2015