OCR Core Maths 3

Past paper questions Exponential Functions

Edited by K V Kumaran

Email: kvkumaran@gmail.com

Phone: 07961319548

Exponentials & Logarithms

- Know that e is a special number in mathematics. It is approximately $2.7182818284\ldots$ and it is irrational (i.e. it can't be expressed as a fraction; similar to π).

- If the base of a logarithm is e then we call it a 'natural logarithm'. Written $\log_e x \equiv \ln x$

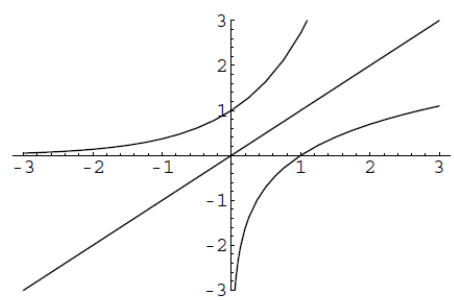
- We already know that logarithms and exponentials are inverses of each other with th relationships

$$\log_{10}(10^x) \equiv x \qquad \text{and} \qquad 10^{\log_{10} x} \equiv x.$$

The same is true for natural logarithms and exponents of e;

$$\ln(e^x) \equiv x \qquad \text{and} \qquad e^{\ln x} \equiv x.$$

- Below is a graph of $y = e^x$ and $y = \ln x$ showing the inverse relationship between the tw (reflecting in $y = x$):



This also shows that you can't 'ln' a negative number and that $\ln 1 = 0$.

- All the laws of logarithms from C2 are true for natural logarithms (e.g. $\ln ab = \ln a + \ln b$ For example make a the subject of the following equation (a few steps missed out):

$$\begin{aligned}
\ln(a-1) - \ln(a+1) &= b \\
\ln\left(\frac{a-1}{a+1}\right) &= b \\
\frac{a-1}{a+1} &= e^b \\
a(1-e^b) &= 1 + e^b \\
a &= \frac{1+e^b}{1-e^b}.
\end{aligned}$$

- You must understand that many physical systems can be modelled by either exponential growth or exponential decay. The most general form is $y = a \times b^x$. If $b > 1$ then the curve represents *exponential growth*. If $b < 1$ then the curve represents *exponential decay*. For example if the number of swine flu sufferers is modelled by $N = 5 \times 7^t$, where t is time measured in days, then find the amount of time for 2 billion people to have caught the disease. We need to solve $2 \times 10^9 = 5 \times 7^t$. So

  $$\frac{2 \times 10^9}{5} = 7^t \quad \Rightarrow \quad \log\left(\frac{2 \times 10^9}{5}\right) = t \log 7 \quad \Rightarrow \quad t = 10.2 \text{ days! (to 3 s.f.).}$$

  (Cue dramatic music...)

- Any exponential relationship $y = a \times b^x$ can be converted to an exponential form using e. This is useful because to differentiate exponential relationships they have to be of the form $y = a \times e^{kt}$. This is done using the powerful statement that (something $\equiv e^{\ln \text{something}}$), so

  $$\begin{aligned} y &= a \times b^x \\ y &= a \times e^{\ln(b^x)} \\ y &= a \times e^{x \ln b} && \text{(by `log law' } \log(a^n) = n \log a) \\ y &= a \times e^{kx}, && \text{(where } k = \ln b). \end{aligned}$$

- An exponential can never equal zero (see graph above). Therefore if you have an equation with lots of exponential 'bits' that you can factorise out, then you are allowed to divide through (in a way that is forbidden with trig functions). For example if $2x^2e^{2x} + 3xe^{2x} - 2e^{2x} = 0$, factorise out the e^{2x} to get $e^{2x}(2x^2 + 3x - 2) = 0$. Divide by e^{2x} to get $2x^2 + 3x - 2 = 0$ which solves to $x = \frac{1}{2}$ or $x = -2$.

- To differentiate an exponential the basic building block is

  $$y = e^x \qquad \Rightarrow \qquad \frac{dy}{dx} = e^x.$$

  That is *why* 'e' is so important; it gives us the exponential that differentiates to itself. Combined with the chain/product/quotient rule (below) we can build on this starting point. (Some students think that if $y = e^x$, then $\frac{dy}{dx} = xe^{x-1}$. Do not be one of them! Exponentials are fundamentally different to polynomials.)

- To differentiate a natural logarithm the basic building block is

  $$y = \ln x \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{x}.$$

1.

The mass, m grams, of a substance at time t years is given by the formula

$$m = 180e^{-0.017t}.$$

(i) Find the value of t for which the mass is 25 grams. [3]

(ii) Find the rate at which the mass is decreasing when $t = 55$. [3]

Q3 June 2005

2.

(a)

t	0	10	20
X	275	440	

The quantity X is increasing exponentially with respect to time t. The table above shows values of X for different values of t. Find the value of X when $t = 20$. [3]

(b) The quantity Y is decreasing exponentially with respect to time t where

$$Y = 80e^{-0.02t}.$$

(i) Find the value of t for which $Y = 20$, giving your answer correct to 2 significant figures. [3]

(ii) Find by differentiation the rate at which Y is decreasing when $t = 30$, giving your answer correct to 2 significant figures. [3]

Q6 Jan 2006

3.

A substance is decaying in such a way that its mass, m kg, at a time t years from now is given by the formula

$$m = 240e^{-0.04t}.$$

(i) Find the time taken for the substance to halve its mass. [3]

(ii) Find the value of t for which the mass is decreasing at a rate of 2.1 kg per year. [4]

Q5 June 2007

4.

It is claimed that the number of plants of a certain species in a particular locality is doubling every 9 years. The number of plants now is 42. The number of plants is treated as a continuous variable and is denoted by N. The number of years from now is denoted by t.

(i) Two equivalent expressions giving N in terms of t are

$$N = A \times 2^{kt} \qquad \text{and} \qquad N = Ae^{mt}.$$

Determine the value of each of the constants A, k and m. [4]

(ii) Find the value of t for which $N = 100$, giving your answer correct to 3 significant figures. [2]

(iii) Find the rate at which the number of plants will be increasing at a time 35 years from now. [3]

Q7 June 2008

5.

The mass, M grams, of a certain substance is increasing exponentially so that, at time t hours, the mass is given by

$$M = 40e^{kt},$$

where k is a constant. The following table shows certain values of t and M.

t	0	21	63
M		80	

(i) In either order,

(a) find the values missing from the table, **[3]**

(b) determine the value of k. **[2]**

(ii) Find the rate at which the mass is increasing when $t = 21$. **[3]**

Q5 Jan 2009

6.

It is given that $p = e^{280}$ and $q = e^{300}$.

(i) Use logarithm properties to show that $\ln\left(\frac{ep^2}{q}\right) = 261$. **[3]**

(ii) Find the smallest integer n which satisfies the inequality $5^n > pq$. **[3]**

Q2 June 2012

7.

(a) Leaking oil is forming a circular patch on the surface of the sea. The area of the patch is increasing at a rate of 250 square metres per hour. Find the rate at which the radius of the patch is increasing at the instant when the area of the patch is 1900 square metres. Give your answer correct to 2 significant figures. **[4]**

(b) The mass of a substance is decreasing exponentially. Its mass now is 150 grams and its mass, m grams, at a time t years from now is given by

$$m = 150e^{-kt},$$

where k is a positive constant. Find, in terms of k, the number of years from now at which the mass will be decreasing at a rate of 3 grams per year. **[3]**

Q7 Jan 2010

8.

An experiment involves two substances, Substance 1 and Substance 2, whose masses are changing. The mass, M_1 grams, of Substance 1 at time t hours is given by

$$M_1 = 400e^{-0.014t}.$$

The mass, M_2 grams, of Substance 2 is increasing exponentially and the mass at certain times is shown in the following table.

t (hours)	0	10	20
M_2 (grams)	75	120	192

A critical stage in the experiment is reached at time T hours when the masses of the two substances are equal.

(i) Find the rate at which the mass of Substance 1 is decreasing when $t = 10$, giving your answer in grams per hour correct to 2 significant figures. **[3]**

(ii) Show that T is the root of an equation of the form $e^{kt} = c$, where the values of the constants k and c are to be stated. **[5]**

(iii) Hence find the value of T correct to 3 significant figures. **[2]**

Q8 June 2011

9.

(i) Substance A is decaying exponentially and its mass is recorded at regular intervals. At time t years, the mass, M grams, of substance A is given by

$$M = 40e^{-0.132t}.$$

(a) Find the time taken for the mass of substance A to decrease to 25% of its value when $t = 0$. **[3]**

(b) Find the rate at which the mass of substance A is decreasing when $t = 5$. **[3]**

(ii) Substance B is also decaying exponentially. Initially its mass was 40 grams and, two years later, its mass is 31.4 grams. Find the mass of substance B after a further year. **[3]**

Q7 Jan 2012

10.

The mass, m grams, of a substance is increasing exponentially so that the mass at time t hours is given by

$$m = 250e^{0.021t}.$$

(i) Find the time taken for the mass to increase to twice its initial value, and deduce the time taken for the mass to increase to 8 times its initial value. **[3]**

(ii) Find the rate at which the mass is increasing at the instant when the mass is 400 grams. **[3]**

Q4 Jan 2013

11.

(a) The mass, M grams, of a substance at time t years is given by

$$M = 58e^{-0.33t}.$$

Find the rate at which the mass is decreasing at the instant when $t = 4$. Give your answer correct to 2 significant figures. **[3]**

(b) The mass of a second substance is increasing exponentially. The initial mass is 42.0 grams and, 6 years later, the mass is 51.8 grams. Find the mass at a time 24 years after the initial value. **[4]**

Q5 June 2014