Edexcel
New GCE A Level Maths workbook
Factor Theorem

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Algebra and functions

Simple algebraic division; use of the *Factor Theorem* and the *Remainder Theorem*.

- Only division by $(x + a)$ or $(x - a)$ will be required.
- Students should know that if

  $f(x) = 0$ when $x = a$, then $(x - a)$ is a factor of $f(x)$.
- Students may be required to factorise cubic expressions such as

  $x^3 + 3x^2 - 4$ and $6x^3 + 11x^2 - x - 6$.

Simplifying Algebraic Fractions

Algebraic fractions can be simplified by division.

Example 1: Simplify $\frac{3x^3 + 5x^2 - 2x}{x}$

$$= \frac{3x^3}{x} + \frac{5x^2}{x} - \frac{2x}{x}$$

$$= 3x^2 + 5x - 2$$

Example 2: Simplify $\frac{6x^4 - 5x^3 + 2x^2}{-2x}$

$$= \frac{6x^4}{-2x} - \frac{5x^3}{-2x} + \frac{2x^2}{-2x}$$

$$= -3x^3 + \frac{5x^2}{2} - x$$

Factorizing to Cancel

Sometimes you need to factorise before you can simplify

Example 1: Simplify $\frac{x^2 + 7x + 12}{x + 3}$

$$= \frac{(x + 3)(x + 4)}{x + 3}$$

$$= x + 4$$

Example 2: Simplify $\frac{2x^2 + 5x - 12}{2x^2 - 7x + 6}$

$$= \frac{(2x - 3)(x + 4)}{(x - 2)(2x - 3)}$$

$$= \frac{x + 4}{x - 2}$$

Dividing a Polynomial by (x±p)

You divide polynomials in the same way as you perform long division.

Example 1: Divide $6x^3 + 28x^2 - 7x + 15$ by $(x + 5)$

$$
\begin{array}{r}
6x^2 - 2x + 3 \\
= (x+5)\overline{)\,6x^3 + 28x^2 - 7x + 15} \\
- \underline{6x^3 + 30x^2} \qquad\qquad \\
-2x^2 - 7x \qquad \\
- \underline{-2x^2 - 10x} \qquad \\
3x + 15 \\
- \underline{3x + 15} \\
0
\end{array}
$$

$$\therefore \frac{6x^3 + 28x^2 - 7x + 15}{x + 5} = 6x^2 - 2x + 3$$

Example 2: Divide $-5x^3 - 27x^2 + 23x + 30$ *by* $(x + 6)$

$$
\begin{array}{r}
-5x^2 + 3x + 5 \\
= (x+6)\overline{)\,-5x^3 - 27x^2 + 23x + 30} \\
- \quad -5x^3 - 30x^2 \qquad\qquad \\
3x^2 + 23x \qquad \\
- \quad 3x^2 + 18x \qquad \\
5x + 30 \\
- \quad 5x + 30 \\
0
\end{array}
$$

$$\therefore \frac{-5x^3 - 27x^2 + 23x + 30}{x + 6} = -5x^2 + 3x + 5$$

Finding the Remainder

After a division of a polynomial, if there is anything left then it is called the remainder. The answer is called the quotient. If there is no remainder then what you are dividing by is said to be a factor.

Example 1: Divide $2x^3 - 5x^2 - 16x + 10$ by $(x - 4)$ and state the remainder and quotient

$$\begin{array}{rl} & 2x^2 + 3x - 4 \\ = (x-4) & \overline{)\,2x^3 - 5x^2 - 16x + 10} \\ - & \underline{2x^3 - 8x^2} \\ & 3x^2 - 16x \\ - & \underline{3x^2 - 12x} \\ & -4x + 10 \\ - & \underline{-4x + 16} \\ & -6 \end{array}$$

$\therefore$ The remainder is -6 and the quotient is $2x^2 + 3x - 4$

Example 2: Divide $2x^3 + 9x^2 + 25$ by $(x + 5)$ and state the remainder and quotient

$$\begin{array}{rl} & 2x^2 - x + 5 \\ = (x+5) & \overline{)\,2x^3 + 9x^2 + 0x + 25} \\ - & 2x^3 + 10x^2 \\ & -x^2 + 0x \\ - & -x^2 - 5x \\ & 5x + 25 \\ - & 5x + 25 \\ & 0 \end{array}$$

Remember to use 0x as there is no x term.

$\therefore$ The remainder is 0 and the quotient is $2x^2 - x + 5$

Factor Theorem

Notes:

Factor Theorem: $(x - a)$ is a factor of a polynomial $f(x)$ if $f(a) = 0$.

Extended version of the factor theorem:

$(ax + b)$ is a factor of a polynomial $f(x)$ if $f\left(\frac{-b}{a}\right) = 0$.

To find out the factors of a polynomial you can quickly substitute in values of x to see which give you a value of zero.

Example 1: Given $f(x) = x^3 + x^2 - 4x - 4$. Use the factor theorem to find a factor

$$f(x) = x^3 + x^2 - 4x - 4$$

$$f(1) = (1)^3 + (1)^2 - (4 \times 1) - 4$$

$$= -6 \qquad \therefore (x - 1) \text{ is not a factor}$$

$$f(2) = (2)^3 + (2)^2 - (4 \times 2) - 4$$

$$= 0 \qquad \therefore (x - 2) \text{ is a factor}$$

To go from the x value to the factor simply put it into a bracket and change the sign.

Remember this:-

$(x + 3)(x - 2) = 0$

$x = -3$ and $x = 2$

We are now going in reverse by putting the x value back into the brackets

Example 2: Given $f(x) = 3x^2 + 8x^2 + 3x - 2$. Factorise fully the given function

$$f(x) = 3x^3 + 8x^2 + 3x - 2$$

$$f(1) = 3 \times (1)^3 + 8 \times (1)^2 + (3 \times 1) - 2$$
$$= 12 \qquad \therefore (x - 1) \text{ is not a factor}$$

$$f(2) = 3 \times (2)^3 + 8 \times (2)^2 + (3 \times 2) - 2$$
$$= 60 \qquad \therefore (x - 2) \text{ is not a factor}$$

$$f(-1) = 3 \times (-1)^3 + 8 \times (-1)^2 + (3 \times -1) - 2$$
$$= 0 \qquad \therefore (x + 1) \text{ is a factor}$$

To fully factorise it we now need to find the quotient. We do this by dividing by the factor we have just found.

$$3x^2 + 5x - 2$$
$$= (x + 1)\overline{\big)\, 3x^3 + 8x^2 + 3x - 2}$$
$$-\ \underline{3x^3 + 3x^2}$$
$$5x^2 + 3x$$
$$-\ \underline{5x^2 + 5x}$$
$$-2x - 2$$
$$-\ \underline{-2x - 2}$$
$$0$$

$$\therefore (x + 1)(3x^2 + 5x - 2) = (x + 1)(3x - 1)(x + 2)$$

We have factorised the quotient further to get the fully factorised answer

Homework Questions 1 – Simplifying Algebraic Fractions

Simplify the following by division

a) $\dfrac{3x^3 + 2x^2 + x}{x}$

b) $\dfrac{5x^3 + 4x^2 - 2x}{-2x}$

c) $\dfrac{4x^3 + 3x^2 + 7x}{x}$

d) $\dfrac{-8x^4 + 6x^3 - 6x^2}{-2x^2}$

e) $\dfrac{5x^2 + 6x - 3}{x^2}$

f) $\dfrac{7x^3 - 3x^2 + 5x}{x^2}$

g) $\dfrac{3x^5 - 6x^3}{2x^2}$

h) $\dfrac{7x^5 + 3x^2 + 6}{3x^2}$

i) $\dfrac{4x^2 + 6x - 5}{2x}$

j) $\dfrac{12x^3 - 9x^2 + 3x}{3x^2}$

Homework Questions 2 – Factorizing to Cancel

Simplify the following by factorizing first

a) $\dfrac{x^2 + 5x + 6}{x + 2}$

b) $\dfrac{x^2 + 6x - 7}{x - 1}$

c) $\dfrac{x^2 + 2x - 15}{x - 3}$

d) $\dfrac{x^2 + 10x + 24}{x + 6}$

e) $\dfrac{2x^2 - 10x + 12}{x - 3}$

f) $\dfrac{x^2 - 9x + 20}{x - 4}$

g) $\dfrac{x^2 - 49}{x - 7}$

h) $\dfrac{x^2 + 10x + 9}{x + 1}$

i) $\dfrac{4x^2 + 12x + 9}{2x + 3}$

j) $\dfrac{7x^3y - 28xy^3}{x - 2y}$

Homework Questions 3 – Dividing a Polynomial

Divide the following to find the quotient

a) $2x^3 + 3x^2 - 3x - 2$ by $(x + 2)$

b) $3x^3 - 2x^2 - 19x - 6$ by $(x - 3)$

c) $x^3 + 7x^2 - 6x - 72$ by $(x - 3)$

e) $4x^3 + 3x^2 - 25x - 6$ by $(x - 2)$

Homework Questions 4 – Finding the Remainder

Find the remainder by division when the following polynomials are divided by:-

a) $x^2 - 4x + 5$ by $(x - 1)$

b) $2x^3 + 3x^2 - 3x + 2$ by $(x + 2)$

c) $2x^2 + 7x - 3$ by $(2x - 1)$

d) $x^3 - 2x^2 - 7x - 1$ by $(x + 1)$

Homework Questions 5 – Factor Theorem

Use the factor theorem to show that the following are factors, and hence fully factorise.

a) $(x - 3)$ is a factor of $x^3 - 3x^2 - 4x + 12$

b) $(x + 2)$ is a factor of $x^3 - 3x^2 - 6x + 8$

c) $(x + 1)$ is a factor of $x^3 + 8x^2 + 19x + 12$

d) $(x - 2)$ is a factor of $x^3 - 4x^2 - 11x + 30$

e) $(x + 3)$ is a factor of $x^3 + 2x^2 - 5x - 6$

(C2-1.6) Name:

1. (*a*) Use the factor theorem to show that $(x + 4)$ is a factor of $2x^3 + x^2 - 25x + 12$.

(*b*) Factorise $2x^3 + x^2 - 25x + 12$ completely.

2. $$f(x) = 2x^3 + x^2 - 5x + c, \text{ where } c \text{ is a constant.}$$

Given that $f(1) = 0$,

(*a*) find the value of c,

(*b*) factorise $f(x)$ completely,

3. $$f(x) = x^3 + 4x^2 + x - 6.$$

(*a*) Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$.
(*b*) Factorise $f(x)$ completely.

(*c*) Write down all the solutions to the equation

$$x^3 + 4x^2 + x - 6 = 0.$$

4. $$f(x) = 2x^3 - 3x^2 - 39x + 20$$

(*a*) Use the factor theorem to show that $(x + 4)$ is a factor of f (x).

(*b*) Factorise f (x) completely.

5. $$f(x) = 2x^3 - 7x^2 - 10x + 24.$$

(*a*) Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$.

(*b*) Factorise $f(x)$ completely.