Edexcel

New GCE A Level Maths workbook

Sketching graphs, Translating graphs.

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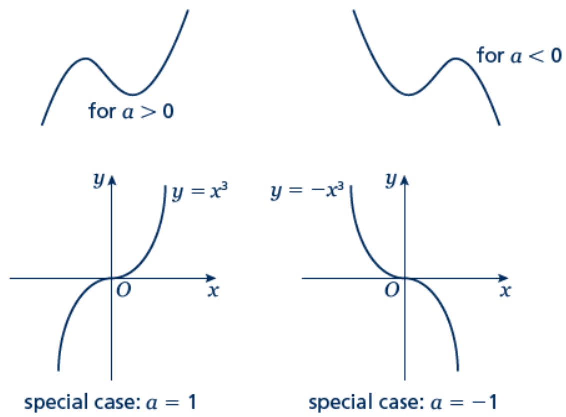
Sketching cubic and reciprocal graphs

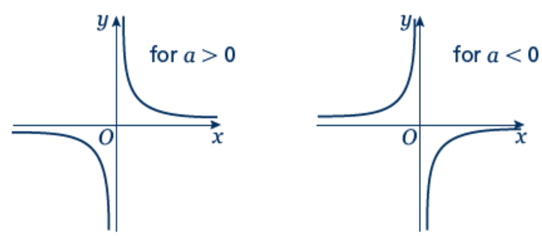
A LEVEL LINKS
Scheme of work: 1e. Graphs – cubic, quartic and reciprocal

Key points

- The graph of a cubic function, which can be written in the form $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$, has one of the shapes shown here.



- The graph of a reciprocal function of the form $y = \frac{a}{x}$ has one of the shapes shown here.

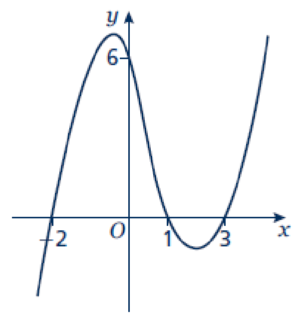


- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the y-axis substitute $x = 0$ into the function.
- To find where the curve intersects the x-axis substitute $y = 0$ into the function.
- Where appropriate, mark and label the asymptotes on the graph.
- Asymptotes are lines (usually horizontal or vertical) which the curve gets closer to but never touches or crosses. Asymptotes usually occur with reciprocal functions. For example, the asymptotes for the graph of $y = \frac{a}{x}$ are the two axes (the lines $y = 0$ and $x = 0$).
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- A double root is when two of the solutions are equal. For example $(x - 3)^2(x + 2)$ has a double root at $x = 3$.
- When there is a double root, this is one of the turning points of a cubic function.

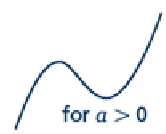
Examples

Example 1 Sketch the graph of $y = (x - 3)(x - 1)(x + 2)$

<table>
<tr><td colspan="2">To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.</td></tr>
<tr><td>When $x = 0$, $y = (0 - 3)(0 - 1)(0 + 2)$
$= (-3) \times (-1) \times 2 = 6$
The graph intersects the y-axis at $(0, 6)$

When $y = 0$, $(x - 3)(x - 1)(x + 2) = 0$
So $x = 3$, $x = 1$ or $x = -2$
The graph intersects the x-axis at
$(-2, 0)$, $(1, 0)$ and $(3, 0)$


</td><td>1 Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. Make sure you get the coordinates the right way around, (x, y).
2 Solve the equation by solving $x - 3 = 0$, $x - 1 = 0$ and $x + 2 = 0$

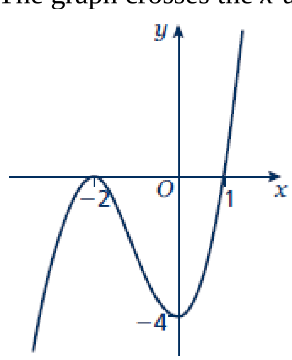
3 Sketch the graph.
$a = 1 > 0$ so the graph has the shape:


</td></tr>
</table>

Example 2 Sketch the graph of $y = (x + 2)^2(x - 1)$

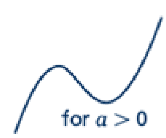
<table>
<tr><td colspan="2">To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.</td></tr>
<tr><td>When $x = 0$, $y = (0 + 2)^2(0 - 1)$
$= 2^2 \times (-1) = -4$
The graph intersects the y-axis at $(0, -4)$

When $y = 0$, $(x + 2)^2(x - 1) = 0$
So $x = -2$ or $x = 1$

$(-2, 0)$ is a turning point as $x = -2$ is a double root.
The graph crosses the x-axis at $(1, 0)$


</td><td>1 Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$.

2 Solve the equation by solving $x + 2 = 0$ and $x - 1 = 0$

3 $a = 1 > 0$ so the graph has the shape:


</td></tr>
</table>

Practice

1 Here are six equations.

A $y = \frac{5}{x}$ **B** $y = x^2 + 3x - 10$ **C** $y = x^3 + 3x^2$

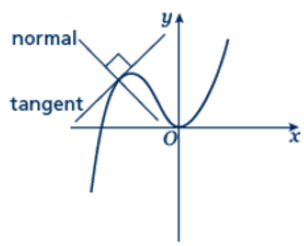
D $y = 1 - 3x^2 - x^3$ **E** $y = x^3 - 3x^2 - 1$ **F** $x + y = 5$

Hint

Find where each of the cubic equations cross the y-axis.

Here are six graphs.

i



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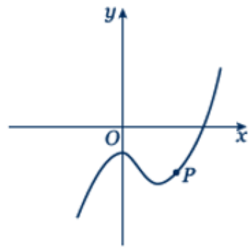

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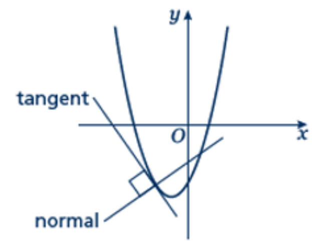

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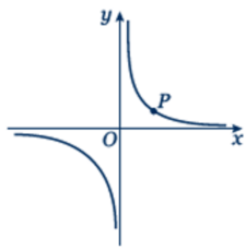

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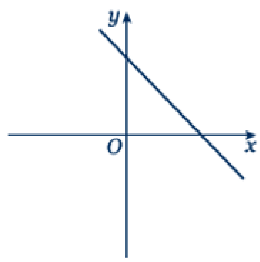

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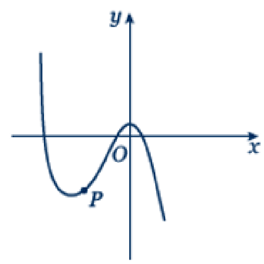


a Match each graph to its equation.

b Copy the graphs ii, iv and vi and draw the tangent and normal each at point P.

Sketch the following graphs

2 $y = 2x^3$

3 $y = x(x - 2)(x + 2)$

4 $y = (x + 1)(x + 4)(x - 3)$

5 $y = (x + 1)(x - 2)(1 - x)$

6 $y = (x - 3)^2(x + 1)$

7 $y = (x - 1)^2(x - 2)$

8 $y = \frac{3}{x}$

Hint: Look at the shape of $y = \frac{a}{x}$ in the second key point.

9 $y = -\frac{2}{x}$

Extend

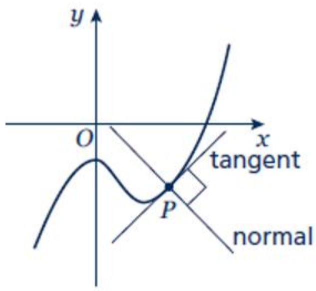
10 Sketch the graph of $y = \frac{1}{x+2}$

11 Sketch the graph of $y = \frac{1}{x-1}$

Answers

1 **a** i – C
ii – E
iii – B
iv – A
v – F
vi – D

b **ii**



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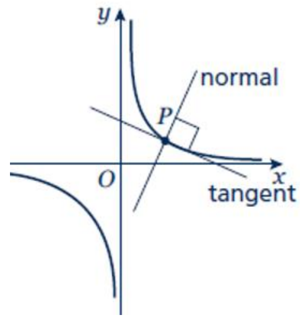

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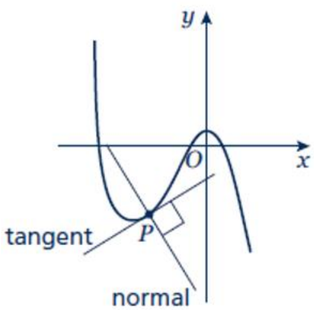

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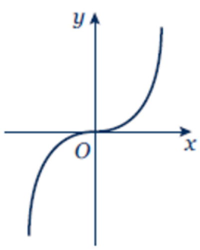

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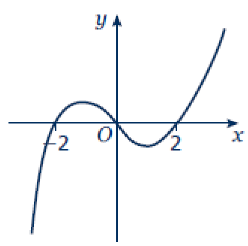

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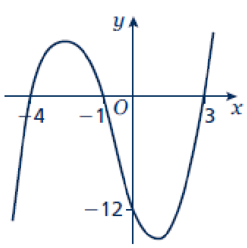

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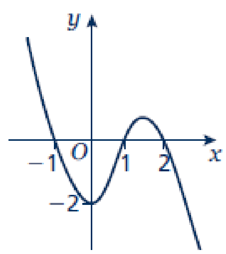

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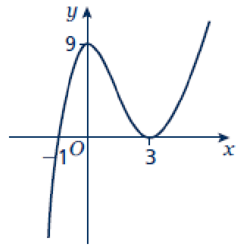

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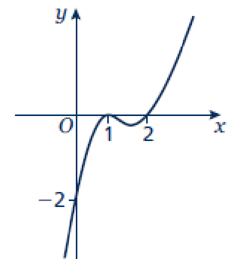

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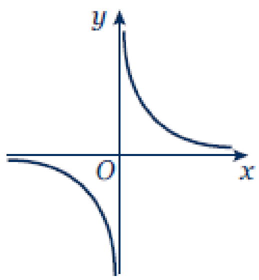

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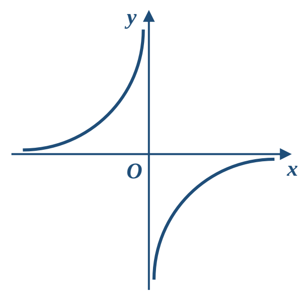

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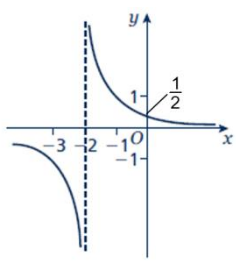

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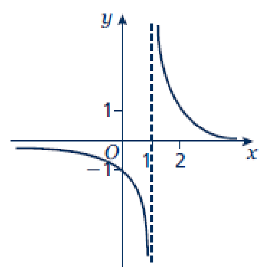


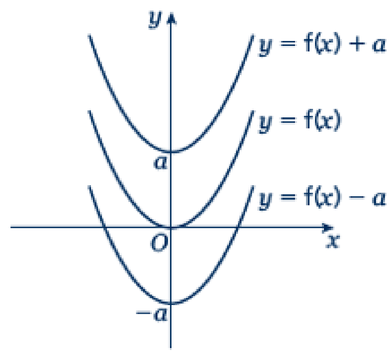
Translating graphs

A LEVEL LINKS
Scheme of work: 1f. Transformations – transforming graphs – f(x) notation

Key points

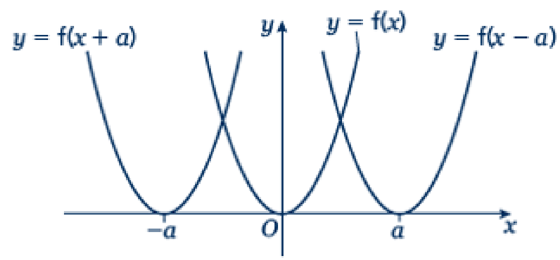
- The transformation $y = \mathrm{f}(x) \pm a$ is a translation of $y = \mathrm{f}(x)$ parallel to the y-axis; it is a vertical translation.

  As shown on the graph,
  - $y = \mathrm{f}(x) + a$ translates $y = \mathrm{f}(x)$ up
  - $y = \mathrm{f}(x) – a$ translates $y = \mathrm{f}(x)$ down.



- The transformation $y = \mathrm{f}(x \pm a)$ is a translation of $y = \mathrm{f}(x)$ parallel to the x-axis; it is a horizontal translation.

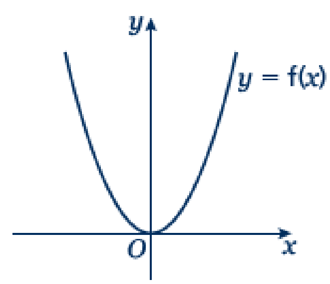
  As shown on the graph,
  - $y = \mathrm{f}(x + a)$ translates $y = \mathrm{f}(x)$ to the left
  - $y = \mathrm{f}(x – a)$ translates $y = \mathrm{f}(x)$ to the right.

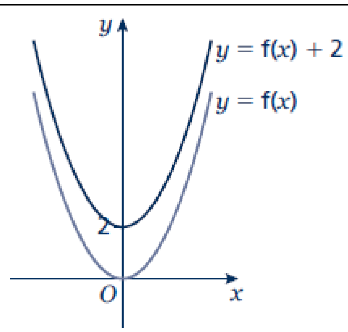


Examples

Example 1 The graph shows the function $y = \mathrm{f}(x)$.

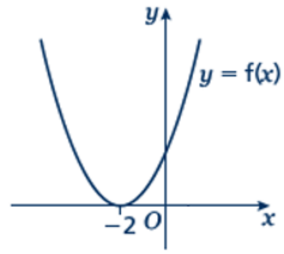
Sketch the graph of $y = \mathrm{f}(x) + 2$.

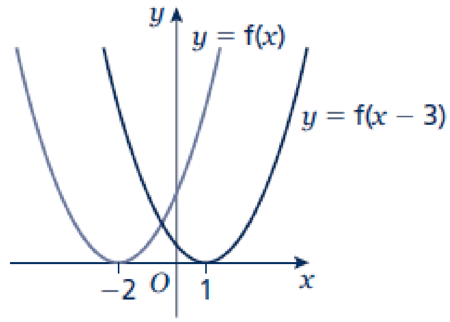


	For the function $y = \mathrm{f}(x) + 2$ translate the function $y = \mathrm{f}(x)$ 2 units up.

Example 2 The graph shows the function $y = f(x)$.

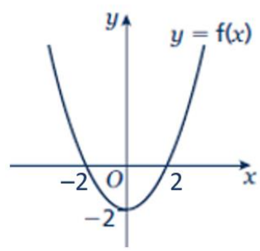
Sketch the graph of $y = f(x - 3)$.



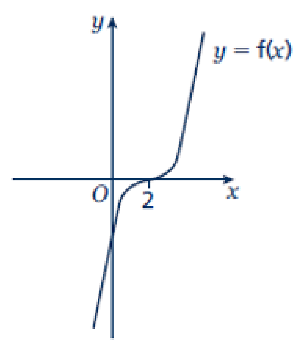
	For the function $y = f(x - 3)$ translate the function $y = f(x)$ 3 units right.

Practice

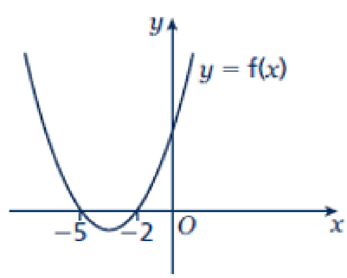
1 The graph shows the function $y = f(x)$.
Copy the graph and on the same axes sketch and label the graphs of $y = f(x) + 4$ and $y = f(x + 2)$.



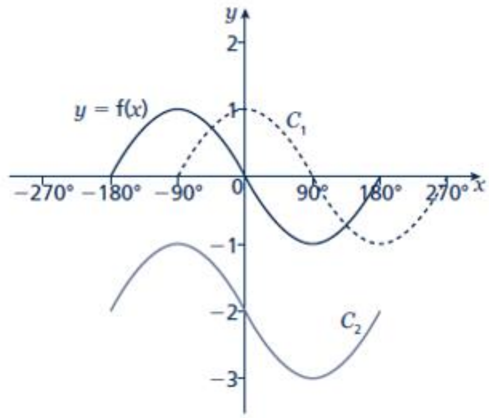
2 The graph shows the function $y = f(x)$.
Copy the graph and on the same axes sketch and label the graphs of $y = f(x + 3)$ and $y = f(x) – 3$.



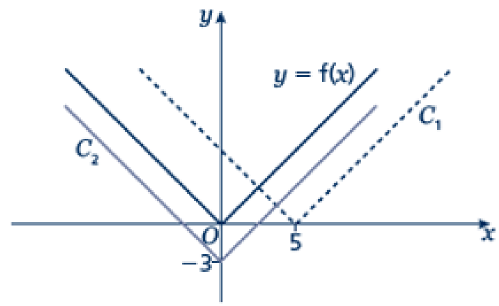
3 The graph shows the function $y = f(x)$.
Copy the graph and on the same axes sketch the graph of $y = f(x – 5)$.



4 The graph shows the function $y = f(x)$ and two tr
Write down the equations of the translated curves C_1 an



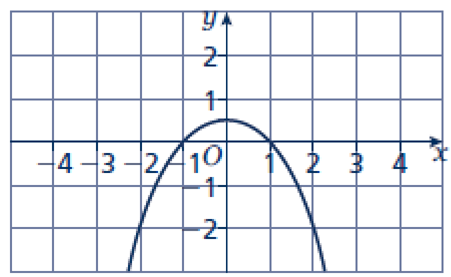
5 The graph shows the function $y = f(x)$ and two transformations of $y = f(x)$, labelled C_1 and C_2.
Write down the equations of the translated curves C_1 and C_2 in function form.



6 The graph shows the function $y = f(x)$.

a Sketch the graph of $y = f(x) + 2$

b Sketch the graph of $y = f(x + 2)$

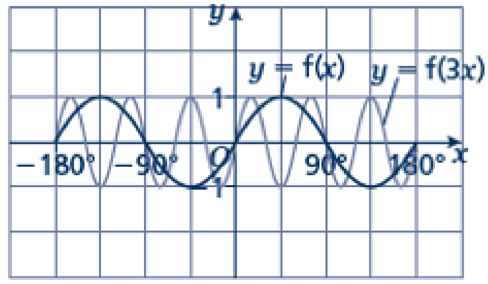


Stretching graphs

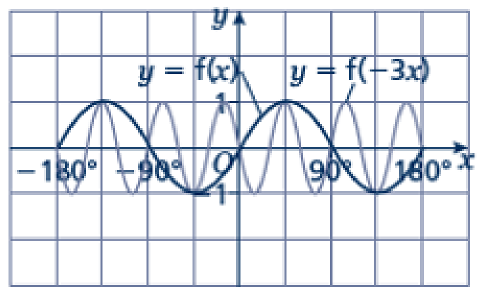
A LEVEL LINKS
Scheme of work: 1f. Transformations – transforming graphs – f(x) notation
Textbook: Pure Year 1, 4.6 Stretching graphs

Key points

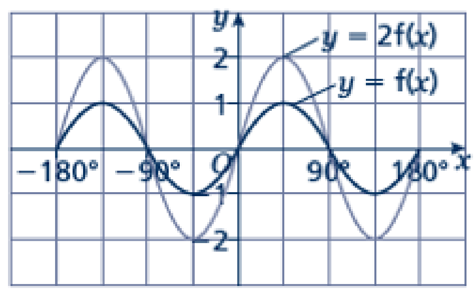
- The transformation y = f(ax) is a horizontal stretch of y = f(x) with scale factor $\frac{1}{a}$ parallel to the x-axis.



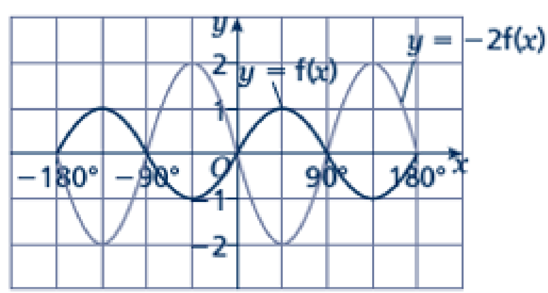
- The transformation y = f($-ax$) is a horizontal stretch of y = f(x) with scale factor $\frac{1}{a}$ paralle to the x-axis and then a reflection in the y-axi



- The transformation y = af(x) is a vertical stretch of y = f(x) with scale factor a parallel t the y-axis.



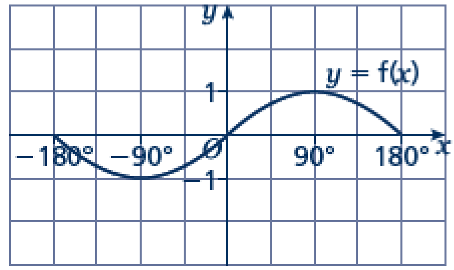
- The transformation y = $-a$f(x) is a vertical stretch of y = f(x) with scale factor a parallel the y-axis and then a reflection in the x-axis.

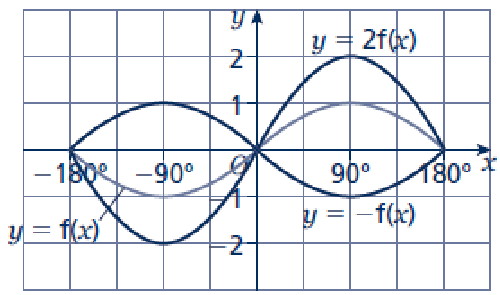


Examples

Example 3 The graph shows the function $y = \mathrm{f}(x)$.

Sketch and label the graphs of
$y = 2\mathrm{f}(x)$ and $y = -\mathrm{f}(x)$.



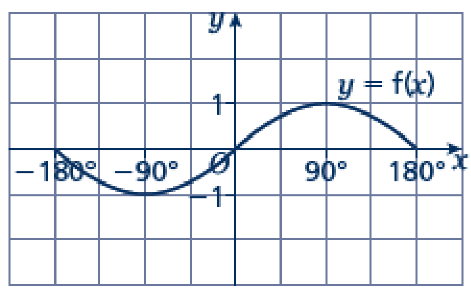


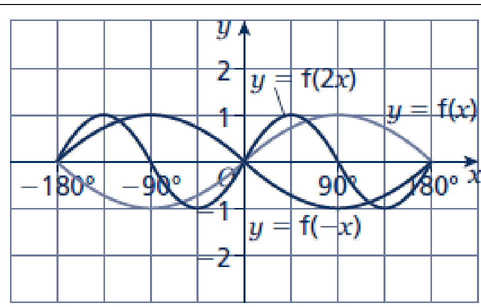
The function $y = 2\mathrm{f}(x)$ is a vertical stretch of $y = \mathrm{f}(x)$ with scale factor 2 parallel to the y-axis.

The function $y = -\mathrm{f}(x)$ is a reflection of $y = \mathrm{f}(x)$ in the x-axis.

Example 4 The graph shows the function $y = \mathrm{f}(x)$.

Sketch and label the graphs of
$y = \mathrm{f}(2x)$ and $y = \mathrm{f}(-x)$.





The function $y = \mathrm{f}(2x)$ is a horizontal stretch of $y = \mathrm{f}(x)$ with scale factor $\frac{1}{2}$ parallel to the x-axis.

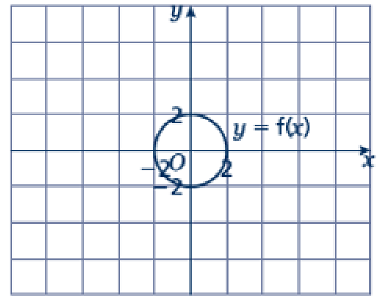
The function $y = \mathrm{f}(-x)$ is a reflection of $y = \mathrm{f}(x)$ in the y-axis.

Practice

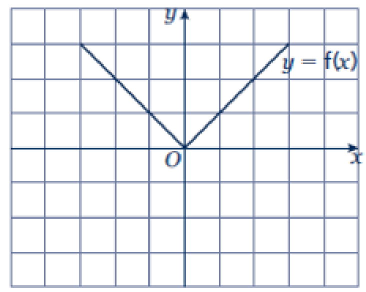
7 The graph shows the function $y = f(x)$.

a Copy the graph and on the same axes sketch and label the graph of $y = 3f(x)$.

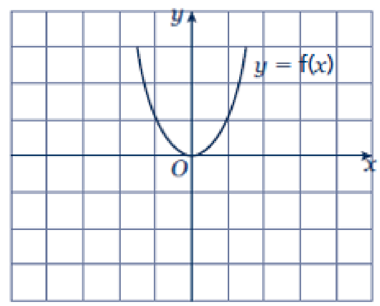
b Make another copy of the graph and on the same axes sketch and label the graph of $y = f(2x)$.



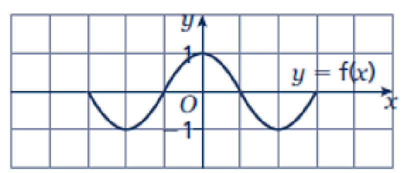
8 The graph shows the function $y = f(x)$. Copy the graph and on the same axes sketch and label the graphs of $y = –2f(x)$ and $y = f(3x)$.



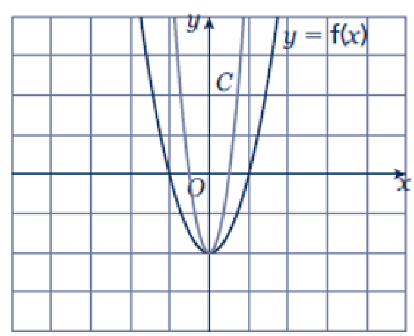
9 The graph shows the function $y = f(x)$. Copy the graph and, on the same axes, sketch and label the graphs of $y = –f(x)$ and $y = f\left(\frac{1}{2}x\right)$.



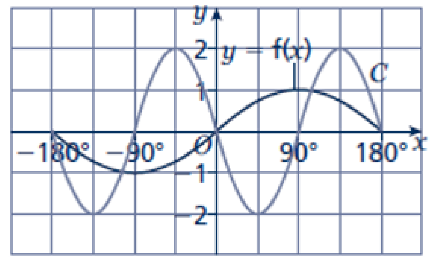
10 The graph shows the function $y = f(x)$. Copy the graph and, on the same axes, sketch the graph of $y = –f(2x)$.



11 The graph shows the function $y = f(x)$ and a transformation, labelled C. Write down the equation of the translated curve C in function form.



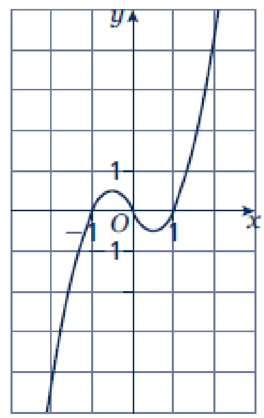
12 The graph shows the function $y = f(x)$ and a transformation labelled C.
Write down the equation of the translated curve C in function form.



13 The graph shows the function $y = f(x)$.

a Sketch the graph of $y = -f(x)$.

b Sketch the graph of $y = 2f(x)$.



Extend

14 **a** Sketch and label the graph of $y = f(x)$, where $f(x) = (x - 1)(x + 1)$.

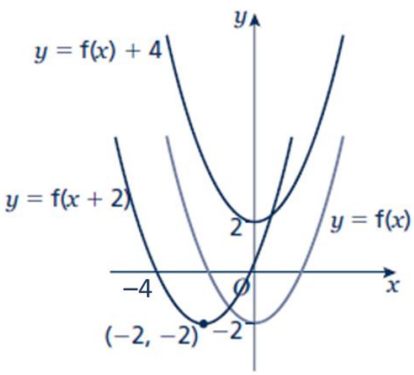
b On the same axes, sketch and label the graphs of $y = f(x) - 2$ and $y = f(x + 2)$.

15 **a** Sketch and label the graph of $y = f(x)$, where $f(x) = -(x + 1)(x - 2)$.

b On the same axes, sketch and label the graph of $y = f\left(-\frac{1}{2}x\right)$.

Answers

1



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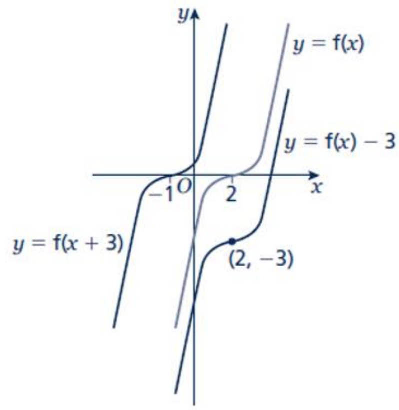

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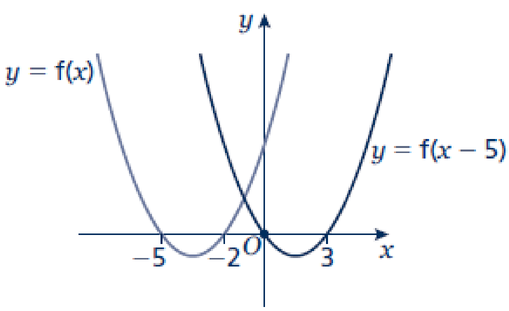
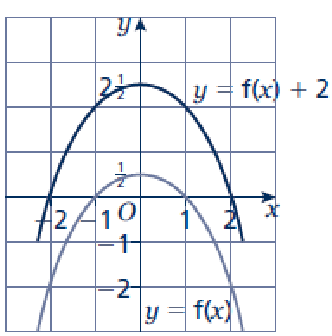


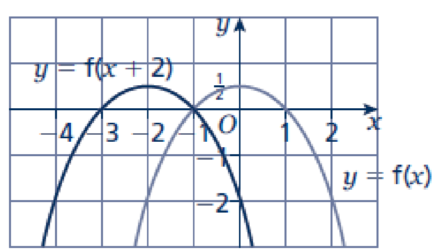
4 C_1: $y = \mathrm{f}(x - 90°)$
C_2: $y = \mathrm{f}(x) - 2$

5 C_1: $y = \mathrm{f}(x - 5)$
C_2: $y = \mathrm{f}(x) - 3$

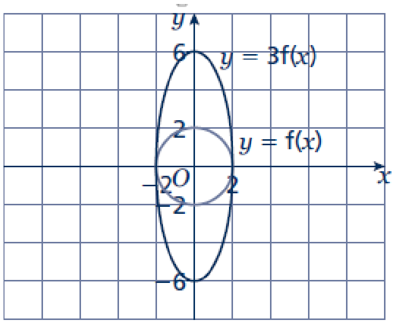
6 a



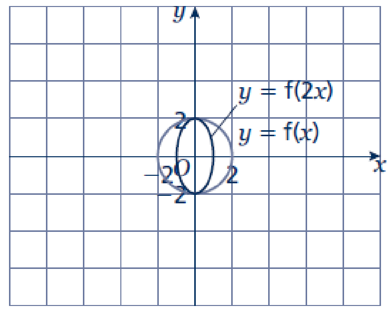
b



7 a



b



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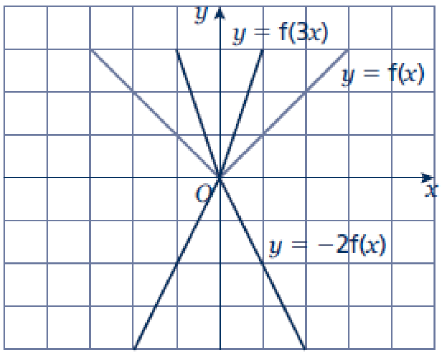

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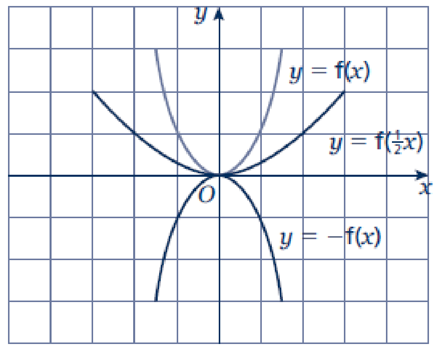
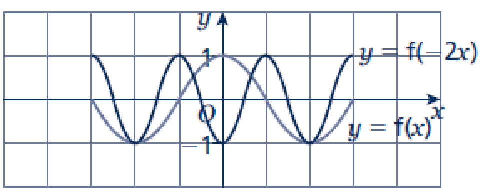


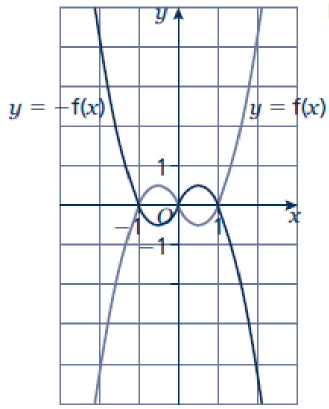
10



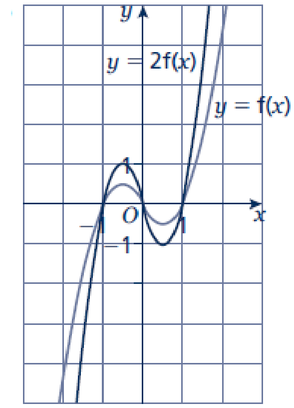
11 $y = f(2x)$

12 $y = –2f(2x)$ or $y = 2f(–2x)$

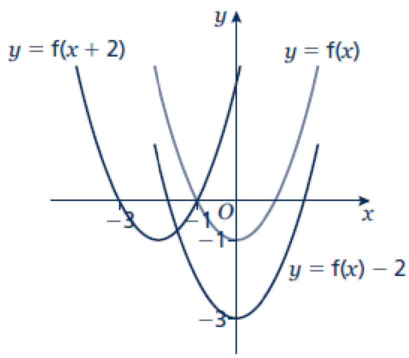
13 a


y
y = −f(x)
y = f(x)
1
O
−1
1
−1
x

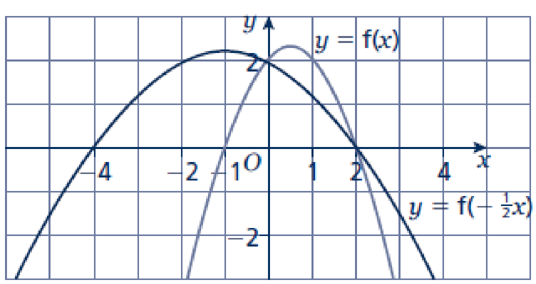
b


y
y = 2f(x)
y = f(x)
1
O
−1
1
−1
x

14


y
y = f(x + 2)
y = f(x)
−3
−1
O
−1
y = f(x) − 2
−3
x

15


y
y = f(x)
2
−4
−2
−1
O
1
2
4
x
y = f(− ½x)
−2

Q1.

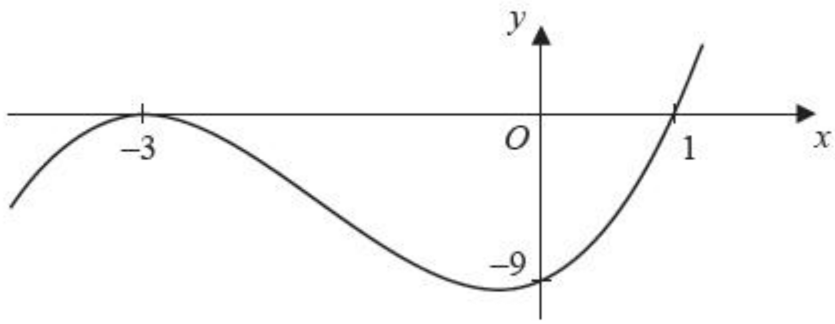


Figure 1

Figure 1 shows a sketch of the curve with equation y = f(x) where

$$f(x) = (x + 3)^2 (x - 1), \qquad x \in \mathbb{R}$$

The curve crosses the x-axis at (1, 0), touches it at (−3, 0) and crosses the y-axis at (0, −9)

(a) In the space below, sketch the curve C with equation $y = f(x + 2)$ and state the coordinates of the points where the curve C meets the x-axis.

(3)

(b) Write down an equation of the curve C.

(1)

(c) Use your answer to part (b) to find the coordinates of the point where the curve C meets the y-axis.

(2)

Q2.

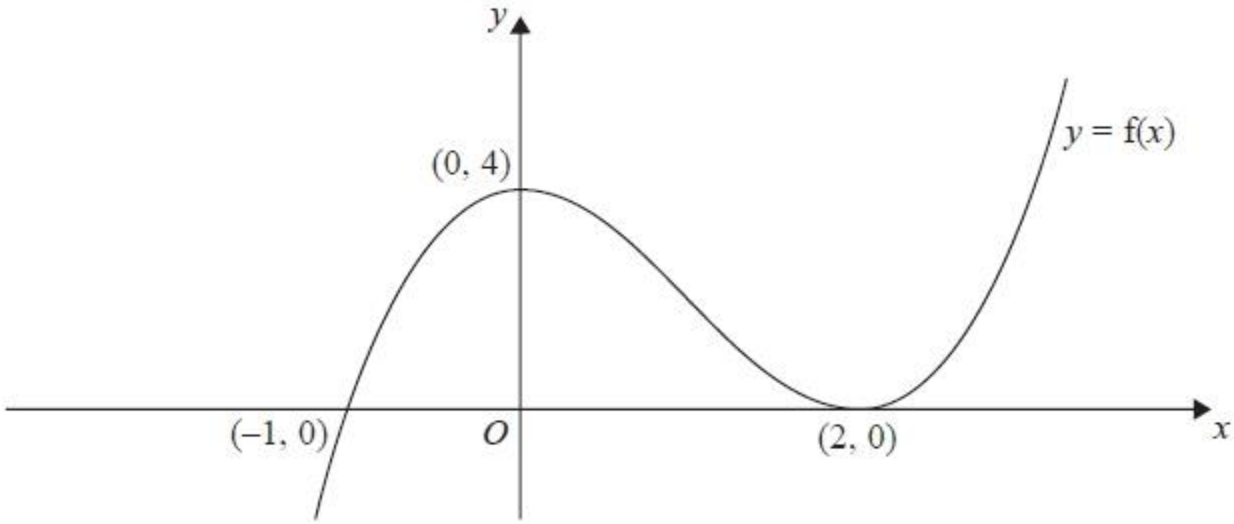


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \text{f}(x)$.

The curve C passes through the point (−1, 0) and touches the x-axis at the point (2, 0).

The curve C has a maximum at the point (0, 4).

(a) The equation of the curve C can be written in the form

$$y = x^3 + ax^2 + bx + c$$

where a, b and c are integers.

Calculate the values of a, b and c.

(5)

(b) Sketch the curve with equation $y = \text{f}(\frac{1}{2})$ in the space provided on page 24

Show clearly the coordinates of all the points where the curve crosses or meets the coordinate axes.

(3)

Q3.

The curve C has equation $y = \frac{3}{x}$ and the line l has equation $y = 2x + 5$.

(a) Sketch the graphs of C and l, indicating clearly the coordinates of any intersections with the axes.

(3)

(b) Find the coordinates of the points of intersection of C and l.

(6)

Q4.

(a) On separate axes sketch the graphs of

(i) $y = -3x + c$, where c is a positive constant,

(ii) $y = \frac{1}{x} + 5$

On each sketch show the coordinates of any point at which the graph crosses the y-axis and the equation of any horizontal asymptote.

(4)

Given that $y = -3x + c$, where c is a positive constant, meets the curve $y = \frac{1}{x} + 5$ at two distinct points,

(b) show that $(5 - c)^2 > 2$

(3)

(c) Hence find the range of possible values for c.

(4)

Q5.

(a) On the axes below, sketch the graphs of

$y = x(x + 2)(3 - x)$

$y = -\frac{2}{x}$

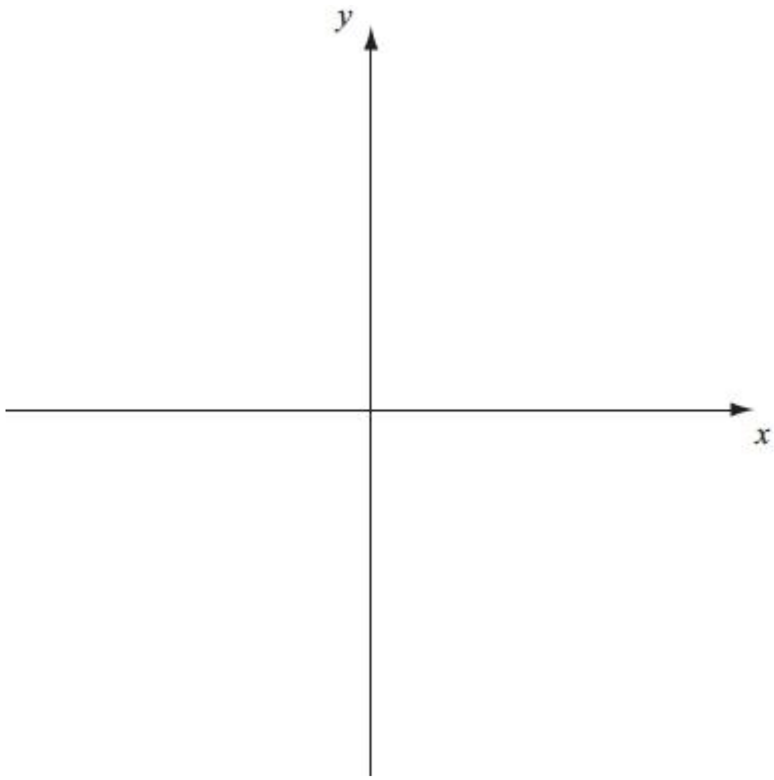
showing clearly the coordinates of all the points where the curves cross the coordinate axes.

(6)

(b) Using your sketch state, giving a reason, the number of real solutions to the equation

$x(x + 2)(3 - x) + \frac{2}{x} = 0$

(2)



Q6.

(a) On the axes below sketch the graphs of

(i) $y = x\,(4 - x)$
(ii) $y = x^2\,(7 - x)$
showing clearly the coordinates of the points where the curves cross the coordinate axes.

(5)

(b) Show that the x-coordinates of the points of intersection of

$y = x(4 - x)$ and $y = x^2(7 - x)$

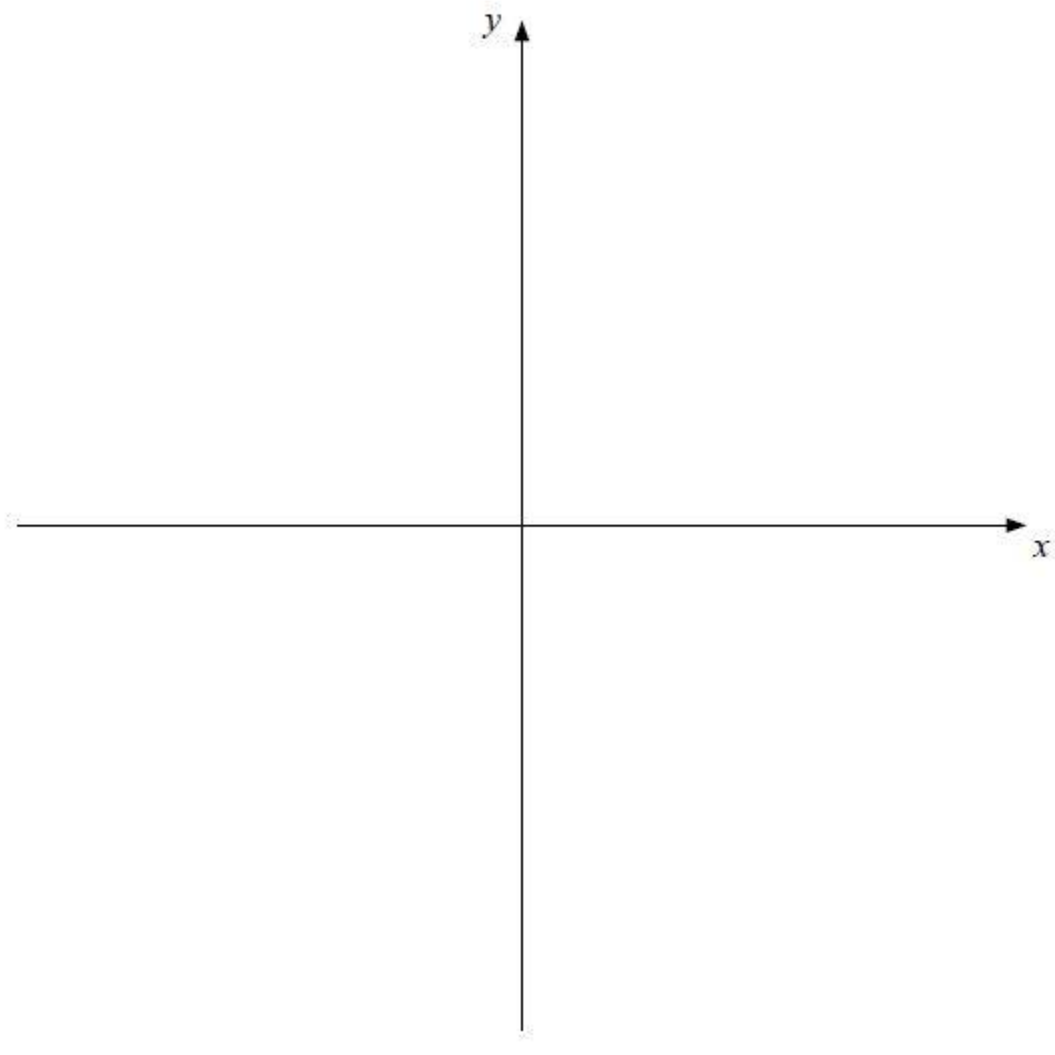
are given by the solutions to the equation $x(x^2 - 8x + 4) = 0$

(3)

The point A lies on both of the curves and the x and y coordinates of A are both positive.

(c) Find the exact coordinates of A, leaving your answer in the form $(p + q\sqrt{3},\ r + s\sqrt{3})$, where p, q, r and s are integers.

(7)



Q7.

(a) Factorise completely $x^3 - 6x^2 + 9x$

(3)

(b) Sketch the curve with equation

$$y = x^3 - 6x^2 + 9x$$

showing the coordinates of the points at which the curve meets the x-axis.

(4)

Using your answer to part (b), or otherwise,

(c) sketch, on a separate diagram, the curve with equation

$$y = (x - 2)^3 - 6(x - 2)^2 + 9(x - 2)$$

showing the coordinates of the points at which the curve meets the x-axis.

(2)

Q8.

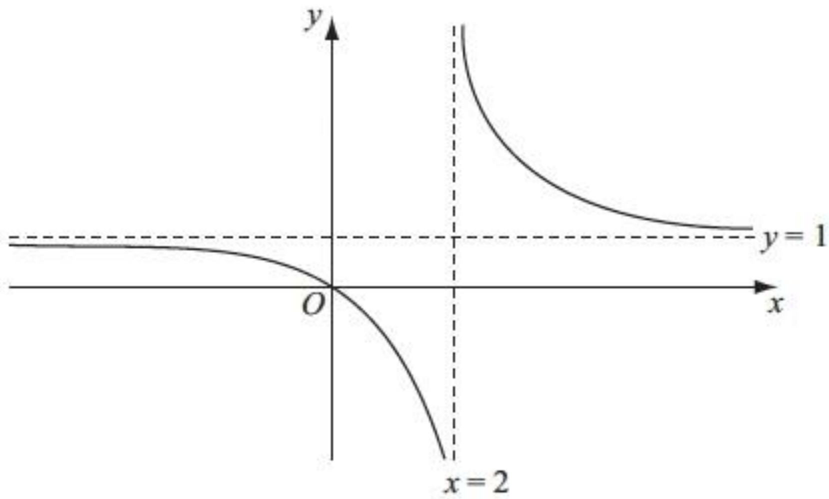


Figure 1

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{x}{x-2}, \qquad x \neq 2$$

The curve passes through the origin and has two asymptotes, with equations $y = 1$ and $x = 2$, as shown in Figure 1.

(a) In the space below, sketch the curve with equation $y = \mathrm{f}(x - 1)$ and state the equations of the asymptotes of this curve.

(3)

(b) Find the coordinates of the points where the curve with equation $y = \mathrm{f}(x - 1)$ crosses the coordinate axes.

(4)

Q9.

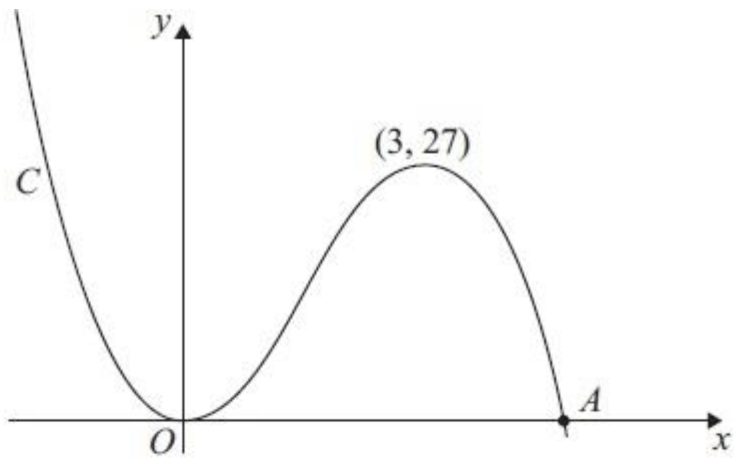


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = f(x)$ where

$$f(x) = x^2(9 - 2x)$$

There is a minimum at the origin, a maximum at the point (3, 27) and C cuts the x-axis at the point A.

(a) Write down the coordinates of the point A.

(1)

(b) On separate diagrams sketch the curve with equation

(i) $y = f(x+3)$
(ii) $y = f(3x)$
On each sketch you should indicate clearly the coordinates of the maximum point and any points where the curves cross or meet the coordinate axes.

(6)

The curve with equation $y = f(x) + k$, where k is a constant, has a maximum point at (3, 10).

(c) Write down the value of k.

(1)

Q10.

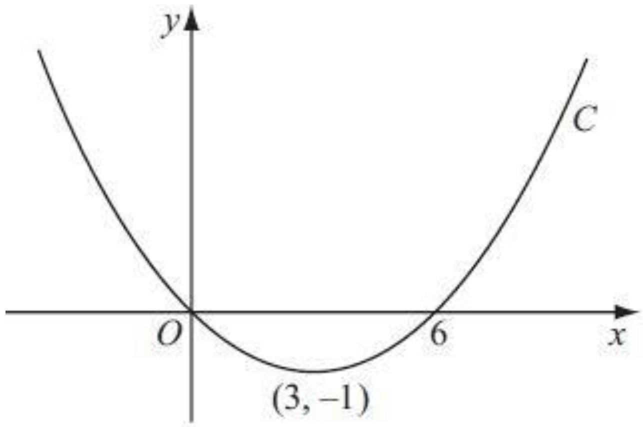


Figure 1

Figure 1 shows a sketch of the curve C with equation y = f (x).
The curve C passes through the origin and through (6, 0).
The curve C has a minimum at the point (3, −1).

On separate diagrams, sketch the curve with equation

(a) y = f (2x),

(3)

(b) y = −f (x),

(3)

(c) y = f (x + p), where p is a constant and $0 < p < 3$.

(4)

Q11.

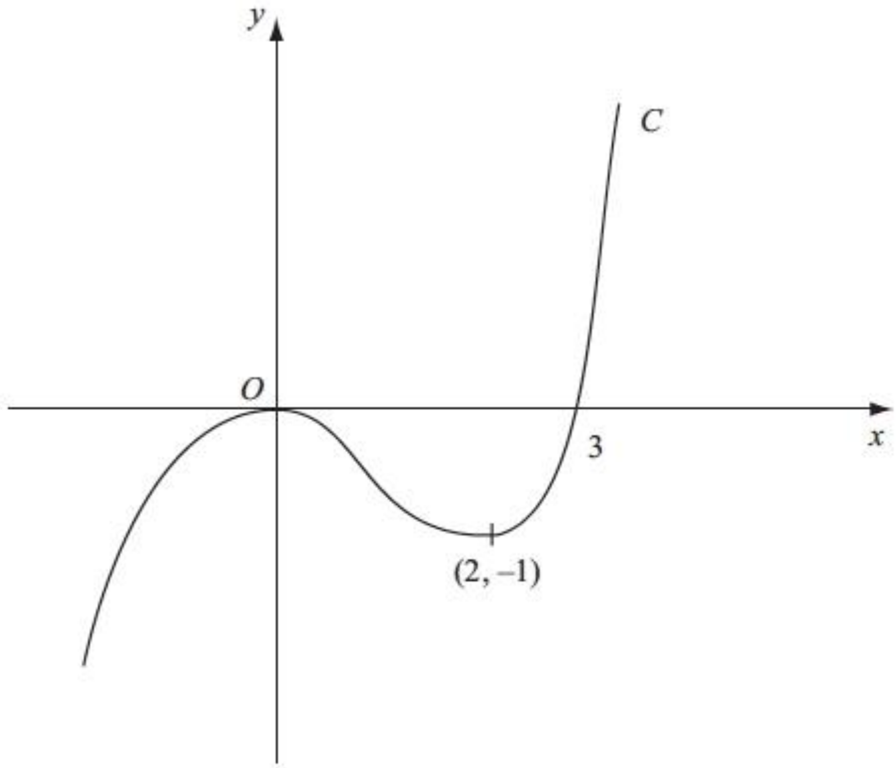


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = f(x)$. There is a maximum at (0, 0), a minimum at (2, −1) and C passes through (3, 0).

On separate diagrams sketch the curve with equation

(a) $y = f(x + 3)$,

(3)

(b) $y = f(-x)$.

(3)

On each diagram show clearly the coordinates of the maximum point, the minimum point and any points of intersection with the x-axis.

Q12.

The point P (1, a) lies on the curve with equation $y = (x + 1)^2(2 - x)$.

(a) Find the value of a.

(1)

(b) On the axes below sketch the curves with the following equations:

(i) $y = (x + 1)^2(2 - x)$,

(ii) $y = \frac{2}{x}$

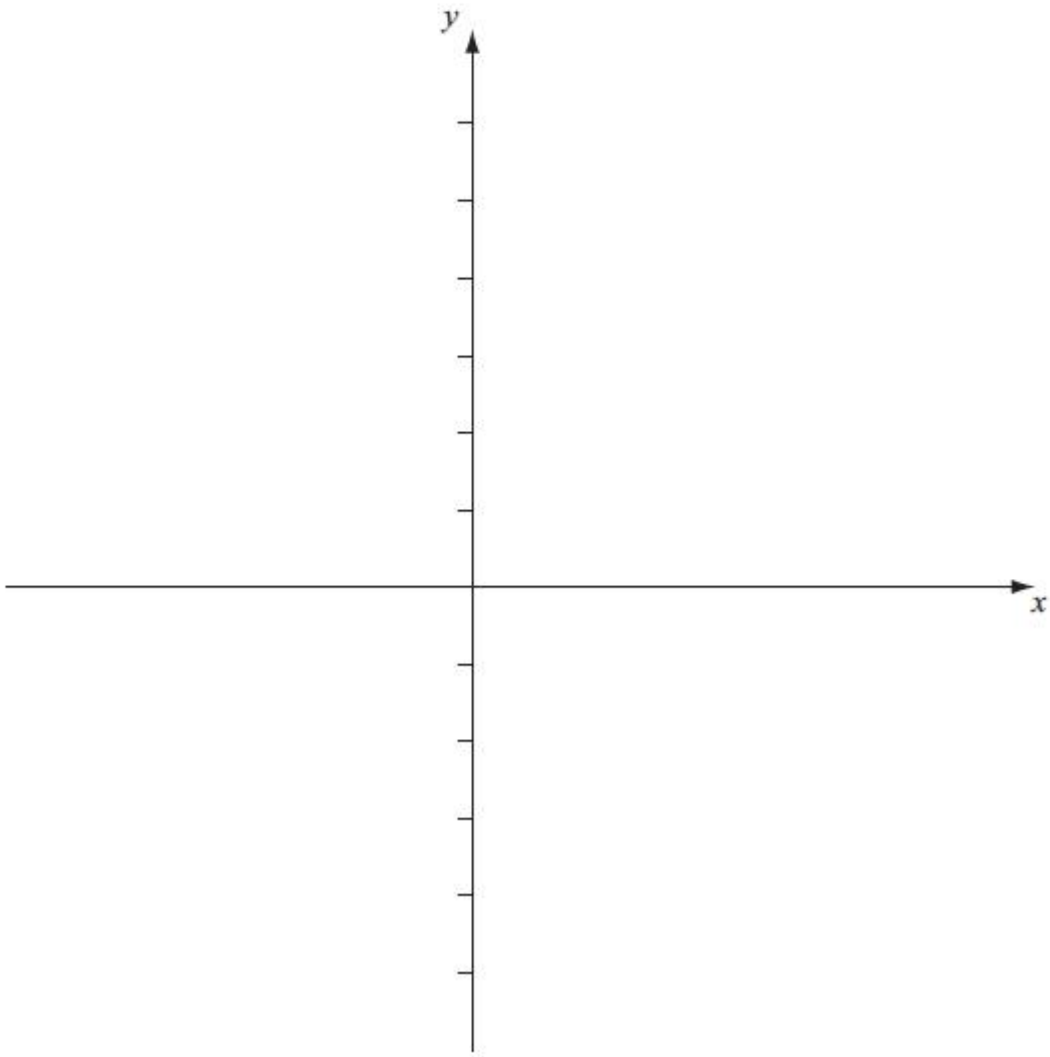
On your diagram show clearly the coordinates of any points at which the curves meet the axes.

(5)

(c) With reference to your diagram in part (b) state the number of real solutions to the equation

$$(x + 1)^2(2 - x) = \frac{2}{x}$$

(1)



Q13.

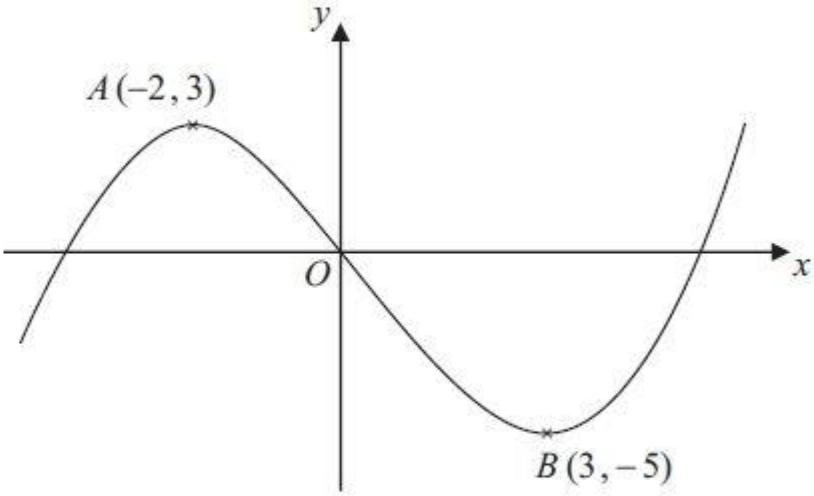


Figure 1

Figure 1 shows a sketch of the curve with equation $y = f(x)$. The curve has a maximum point A at $(-2, 3)$ and a minimum point B at $(3, -5)$.

On separate diagrams sketch the curve with equation

(a) $y = f(x + 3)$

(3)

(b) $y = 2f(x)$

(3)

On each diagram show clearly the coordinates of the maximum and minimum points.

The graph of $y = f(x) + a$ has a minimum at $(3, 0)$, where a is a constant.

(c) Write down the value of a.

(1)

Q14.

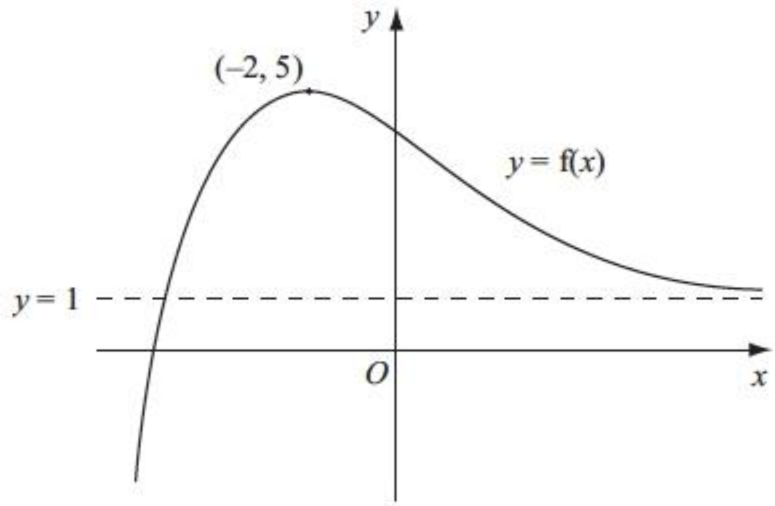


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = f(x)$.

The curve has a maximum point $(-2, 5)$ and an asymptote $y = 1$, as shown in Figure 1.

On separate diagrams, sketch the curve with equation

(a) $y = f(x) + 2$

(2)

(b) $y = 4f(x)$

(2)

(c) $y = f(x + 1)$

(3)

On each diagram, show clearly the coordinates of the maximum point and the equation of the asymptote.

Q15.

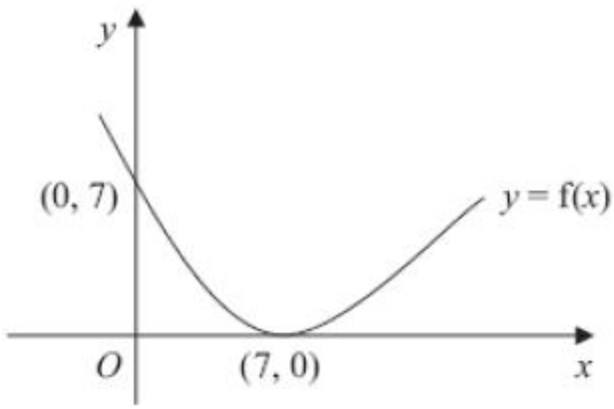


Figure 1

Figure 1 shows a sketch of the curve with equation $y = f(x)$. The curve passes through the point (0, 7) and has a minimum point at (7, 0).

On separate diagrams, sketch the curve with equation

(a) $y = f(x) + 3$,

(3)

(b) $y = f(2x)$.

(2)

On each diagram, show clearly the coordinates of the minimum point and the coordinates of the point at which the curve crosses the y-axis.

Q16.

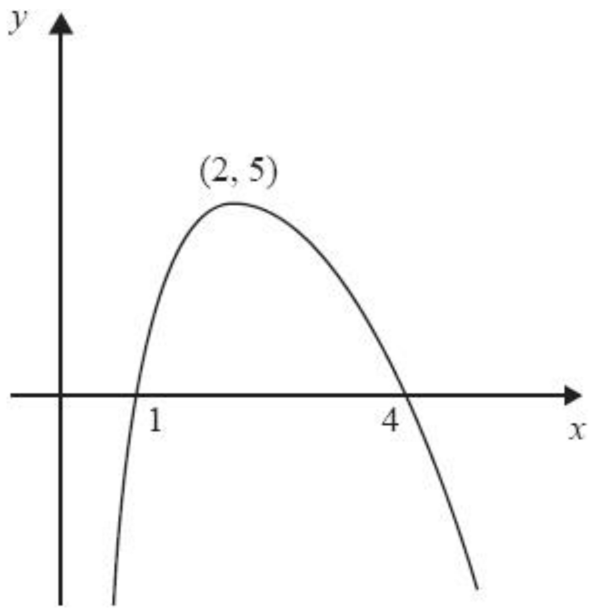


Figure 1

Figure 1 shows a sketch of the curve with equation $y = f(x)$. The curve crosses the x-axis at the points (1, 0) and (4, 0). The maximum point on the curve is (2, 5).
In separate diagrams sketch the curves with the following equations.
On each diagram show clearly the coordinates of the maximum point and of each point at which the curve crosses the x-axis.

(a) $y = 2f(x)$,

(3)

(b) $y = f(-x)$.

(3)

The maximum point on the curve with equation $y = f(x + a)$ is on the y-axis.

(c) Write down the value of the constant a.

(1)

Q17.

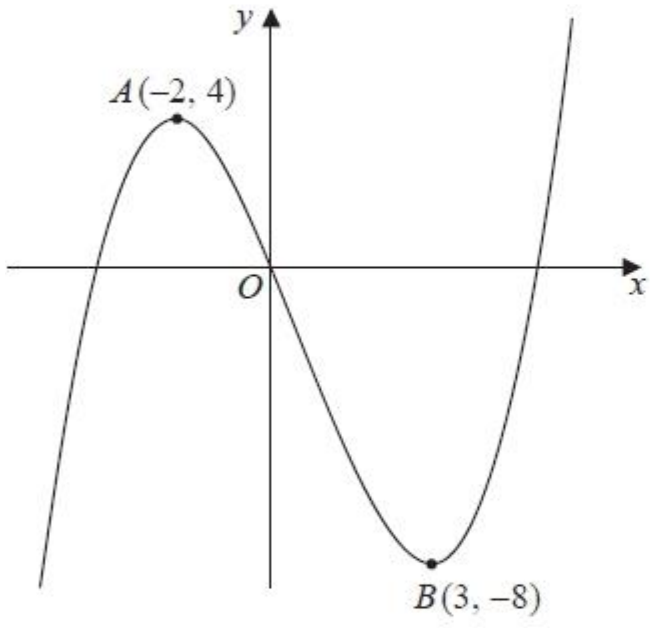


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$. The curve has a maximum point A at (–2, 4) and a minimum point B at (3, –8) and passes through the origin O.

On separate diagrams, sketch the curve with equation

(a) $y = 3\mathrm{f}(x)$,

(2)

(b) $y = \mathrm{f}(x) - 4$

(3)

On each diagram, show clearly the coordinates of the maximum and the minimum points and the coordinates of the point where the curve crosses the y-axis.