Edexcel

New GCE A Level Maths workbook

Circle.

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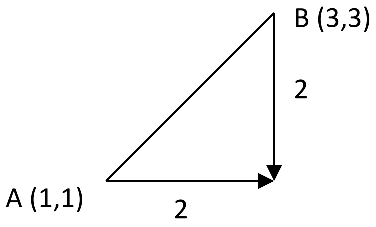
Finding the Midpoint of a Line

To work out the midpoint of line
we need to find the halfway point

Midpoint of AB = (2,2)



The formula for the midpoint is:-

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Where (x_1, y_1) and (x_2, y_2) are 2 given points on the line

Example 1. If A(3,7) and B(11, – 3) Find the midpoint of AB

$$\text{Midpoint of AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$= \left(\frac{3 + 11}{2}, \frac{7 + -3}{2} \right)$$

$$= \left(\frac{14}{2}, \frac{4}{2} \right)$$

$$= (7,2)$$

Diameter of circles are often used in this topic because the midpoint will always be the centre of the circle.

Example 2. *If A(2,3) and B is(5,9) and the centre of the circle. If AC is the diameter of the circle find the coordinates of C*

$$\text{Midpoint of AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$(5, 9) = \left(\frac{2 + x}{2}, \frac{3 + y}{2} \right)$$

$$\therefore\ 5 = \frac{2 + x}{2} \quad \text{and} \quad 9 = \frac{3 + y}{2}$$

$$10 = 2 + x \qquad 18 = 3 + y$$

$$8 = x \qquad 15 = y$$

$$\therefore\ C = (5,15)$$

$$\text{if } x = \frac{10}{4} \quad \text{then} \quad y = -x - 1$$

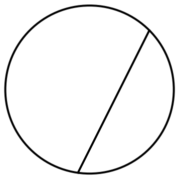
$$y = -\frac{10}{4} - 1$$

$$y = -3{\cdot}5$$

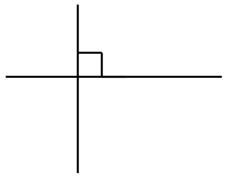
$\therefore$ The Centre of the circle is $(2.5, -3.5)$

Chords and Perpendicular Lines

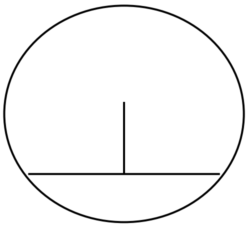
- A chord is a line that passes from one side of a circle to the other but which does not pass through the centre.



- A perpendicular line always cuts at 90°. If it bisects a line then it cuts it exactly in half. It is often called a perpendicular bisector. When questions are talking about this then you need to use the equation of a normal and the midpoints.



- The perpendicular bisector of a chord always passes through the centre of a circle.

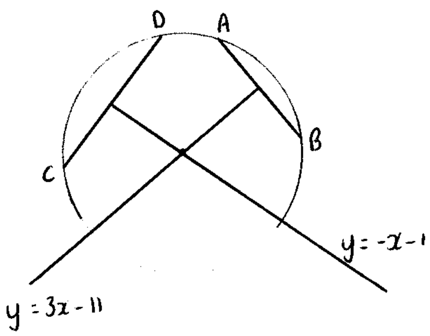


- The key to success is that you always need to draw a sketch so you know what is going on.

Example 1. *The Lines AB and CD are chords of a circle. The line $y = 3x - 11$ is the perpendicular bisector of AB. The line $y = -x - 1$ is the perpendicular bisector of CD. Find the coordinates of the centre of the circle.*

We know the perpendicular bisector of a chord passes through the centre

so the centre of the circle is where the lines meet! So solve simultaneously



$$y = 3x - 11$$

$$y = -x - 1$$

$$\therefore \quad 3x - 11 = -x - 1$$

$$4x - 11 = -1$$

$$4x = 10$$

$$x = \frac{10}{4}$$

if $x = \frac{10}{4}$ then $y = -x - 1$

$$y = -\frac{10}{4} - 1$$

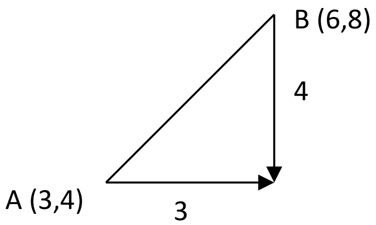
$$y = -3{\cdot}5$$

$\therefore$ The Centre of the circle is $(2.5, \ -3.5)$

Distance Between Two Points

To work out the distance between two points we use Pythagoras

Midpoint of AB = (2,2)



The formula for the distance between two points is:-

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Where (x_1, y_1) and (x_2, y_2) are 2 given points on the line

Example 1. PQ is the diameter of a circle where $p(-1,3)$ and $Q(6,-3)$.

Find the radius of the circle

First we need to remember that Radius = half the Diameter

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$PQ = \sqrt{(6 - -1)^2 + (-3 - 3)^2}$$

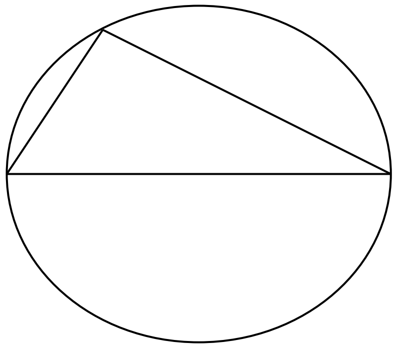
$$PQ = \sqrt{(7^2 + (-6)^2)}$$

$$PQ = \sqrt{85}$$

$$\textbf{Radius} = \frac{\sqrt{85}}{2}$$

Angles in a semicircle

An angle in a semicircle is always 90° when one side of the triangle is the diameter and all 3 sides sit on the circumference of the circle



Example 2. *The points A(2,6) , B(5,7) and C(8, – 2) lie on a circle. Show that ABC is a right angled triangle and find the area of the triangle*

$$\text{Length AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{(5-2)^2 + (7-6)^2}$$

$$AB = \sqrt{3^2 + 1^2}$$

$$AB = \sqrt{10}$$

$$\text{Length BC} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$BC = \sqrt{(8-5)^2 + (-2-7)^2}$$

$$BC = \sqrt{3^2 + (-9)^2}$$

$$BC = \sqrt{90}$$

$$\text{Length AC} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AC = \sqrt{(8-2)^2 + (-2-6)^2}$$

$$AC = \sqrt{6^2 + (-8)^2}$$

$$AC = \sqrt{100}$$

$$AC = 10$$

$\therefore$ Using pythagoras to prove ABC is a right angled triangle

$$AC^2 = AB^2 + BC^2$$

$$10^2 = (\sqrt{10})^2 + (\sqrt{90})^2$$

$$100 = 10 + 90$$

$$100 = 100$$

This proves the triangle is a right angled triangle

$$\text{Area of a triangle} = \frac{1}{2} \times base \times height$$

$$A = \frac{1}{2} \times \text{AB} \times \text{BC}$$

$$A = \frac{1}{2} \times \sqrt{10} \times \sqrt{90}$$

$$A = \frac{1}{2} \times \sqrt{10} \times \sqrt{9} \times \sqrt{10}$$

$$A = 15 \text{ units}^2$$

Equation of a Circle

An equation of a circle is always in the form $(x - a)^2 + (y - b)^2 = r^2$
where r is the radius and (a,b) is the centre of the circle.

Example 1. If a circle has a radius of 7 and a centre at (2,6), what is the equation of the circle?

$(x - a)^2 + (y - b)^2 = r^2$ where a = 2, b = 6, r = 7

$\therefore \quad (x - 2)^2 + (y - 6)^2 = 7^2$

$(x - 2)^2 + (y - 6)^2 = 49$

The equation of the cirlce is $(x - 2)^2 + (y - 6)^2 = 49$

Example 2. Given the equation $(x - 2\sqrt{3})^2 + (y + \sqrt{7})^2 = 144$, find the radius of the centre of the circle.

$(x - a)^2 + (y - b)^2 = r^2$ where $a = 2\sqrt{3}$, $b = -\sqrt{7}$, $r^2 = 144$

$\therefore$ centre is $(2\sqrt{3}, -\sqrt{7})$

$r^2 = 144$

$r = \pm 12$ as radius cannot be negative we can ingnore the negative value

$\therefore r = 12$

Example 3. Prove that (1,2) lies on the circumference of the circle which has the equation

$(x - 2)^2 + (y + 3)^2 = 26$ when x = 1 y = 2

$(1 - 2)^2 + (2 + 3)^2 = 26$

$(-1)^2 + (5)^2 = 26$

$1 + 25 = 26$

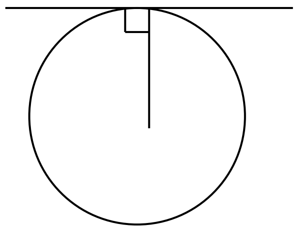
$26 = 26$

$\therefore$ (1,2) lies on the circumference of the circle

Tangents

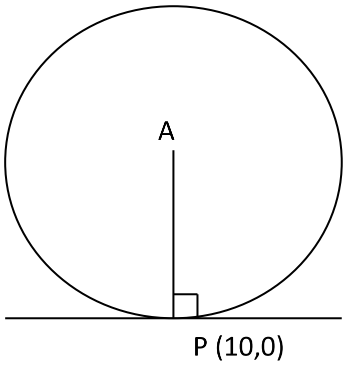
The angle between the tangent and a radius is 90°. A tangent only touches at one point.

This circle theorem is often used in questions as it can relate closely to perpendicular bisectors.



Example 1. The line $4x - 3y - 40 = 0$ touches the circle $(x-2)^2 + (y-6)^2 = 100$ at P(10,0). Show that the radius at P is perpendicular to the line.

This mean the centre A is (2,6)



$$\text{Gradient of AP} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{6-1}{2-10}$$

$$= \frac{6}{-8}$$

$$\therefore \text{ gradient of AP is } -\frac{3}{4}$$

Gradient of tangent

$$4x - 3y - 40 = 0$$

$$4x - 40 = 3y$$

$$\frac{4x-40}{3} = y$$

$$\therefore \text{gradient of tangent is } \frac{4}{3}$$

Using $m_1 \times m_2 = -1$ where $m_1 = -\frac{3}{4}$ and $m_2 = \frac{4}{3}$

$$-\frac{3}{4} \times \frac{4}{3} = -1$$

$\therefore$ lines are perpendicualr

Finding Points of Intersection

If you need to find where a circle meets a line then solve the two equations simultaneously.

Example 1. Find where the line $y = x + 5$ meets the circle $x^2 + (y - 2)^2 = 29$

Substitute $y = x + 5$ into $x^2 + (y - 2)^2 = 29$

$$x^2 + ((x + 5) - 2)^2 = 29$$

$$x^2 + (x + 3)^2 = 29$$

$$x^2 + x^2 + 6x + 9 = 29$$

$$2x^2 + 6x - 20 = 0$$

$$x^2 + 3x - 10 = 0$$

$$(x + 5)(x - 2) = 0$$

$$x = -5 \quad \text{or} \quad x = 2$$

if $x = -5$

$$y = x + 5$$

$$y = -5 + 5$$

$$y = 0$$

$(-5,0)$

if $x = 2$

$$y = x + 5$$

$$y = 2 + 5$$

$$y = 7$$

$(2,7)$

So the line meets the circle at (-5,0) and (2,7).

If you get no solutions when you try and solve two equations then it means the lines do not meet

Q1.

A circle C has centre $(-1, 7)$ and passes through the point $(0, 0)$. Find an equation for C.

(4)

Q2.

The circle C has equation

$$x^2 + y^2 + 4x - 2y - 11 = 0$$

Find

(a) the coordinates of the centre of C,

(2)

(b) the radius of C,

(2)

(c) the coordinates of the points where C crosses the y-axis, giving your answers as simplified surds.

(4)

Q3.

The circle C has equation

$$x^2 + y^2 - 6x + 4y = 12$$

(a) Find the centre and the radius of C.

(5)

The point $P(-1, 1)$ and the point $Q(7, -5)$ both lie on C.

(b) Show that PQ is a diameter of C.

(2)

The point R lies on the positive y-axis and the angle $PRQ = 90°$.

(c) Find the coordinates of R.

(4)

Q4.

The points A and B have coordinates $(-2, 11)$ and $(8, 1)$ respectively.

Given that AB is a diameter of the circle C,

(a) show that the centre of C has coordinates $(3, 6)$,

(1)

(b) find an equation for C.

(4)

(c) Verify that the point $(10, 7)$ lies on C.

(1)

(d) Find an equation of the tangent to C at the point $(10, 7)$, giving your answer in the form $y = mx + c$, where m and c are constants.

(4)

Q5.

The circle C has equation

$$x^2 + y^2 - 10x + 6y + 30 = 0$$

Find

(a) the coordinates of the centre of C,

(2)

(b) the radius of C,

(2)

(c) the y coordinates of the points where the circle C crosses the line with equation $x = 4$, giving your answers as simplified surds.

(3)

Q6.

A circle C with centre at the point $(2, –1)$ passes through the point A at $(4, –5)$.

(a) Find an equation for the circle C.

(3)

(b) Find an equation of the tangent to the circle C at the point A, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(4)

Q7.

The circle C, with centre A, passes through the point P with coordinates $(-9, 8)$ and the point Q with coordinates $(15, -10)$.

Given that PQ is a diameter of the circle C,

(a) find the coordinates of A,

(2)

(b) find an equation for C.

(3)

A point R also lies on the circle C.
Given that the length of the chord PR is 20 units,

(c) find the length of the shortest distance from A to the chord PR. Give your answer as a surd in its simplest form.

(2)

Q8.

The circle C has centre $(3, 1)$ and passes through the point $P(8, 3)$.

(a) Find an equation for C.

(4)

(b) Find an equation for the tangent to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(5)

Q9.

The circle C has centre $A(2,1)$ and passes through the point $B(10, 7)$.

(a) Find an equation for C.

(4)

The line l_1 is the tangent to C at the point B.

(b) Find an equation for l_1.

(4)

The line l_2 is parallel to l_1 and passes through the mid-point of AB.

Given that l_2 intersects C at the points P and Q,

(c) find the length of PQ, giving your answer in its simplest surd form.

(3)

Q10.

The circle C has equation

$$x^2 + y^2 - 20x - 24y + 195 = 0$$

The centre of C is at the point M.

(a) Find

(i) the coordinates of the point M,
(ii) the radius of the circle C.

(5)

N is the point with coordinates $(25, 32)$.

(b) Find the length of the line MN.

(2)

The tangent to C at a point P on the circle passes through point N.

(c) Find the length of the line NP.

(2)

Q11.

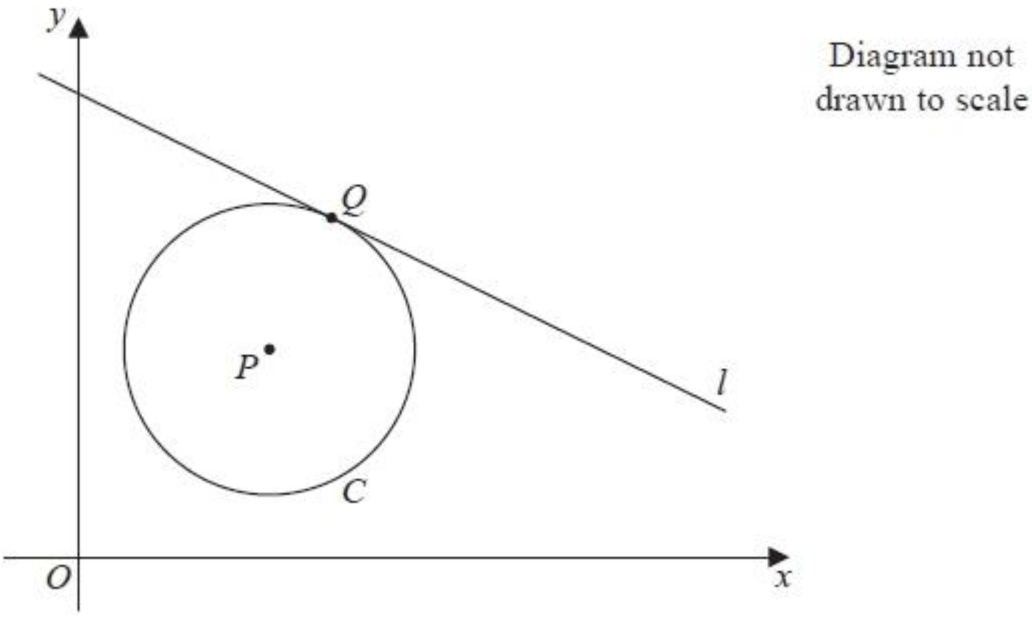


Figure 2

The circle C has centre $P(7, 8)$ and passes through the point $Q(10, 13)$, as shown in Figure 2.

(a) Find the length PQ, giving your answer as an exact value.

(2)

(b) Hence write down an equation for C.

(2)

The line l is a tangent to C at the point Q, as shown in Figure 2.

(c) Find an equation for l, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(4)

Q12.

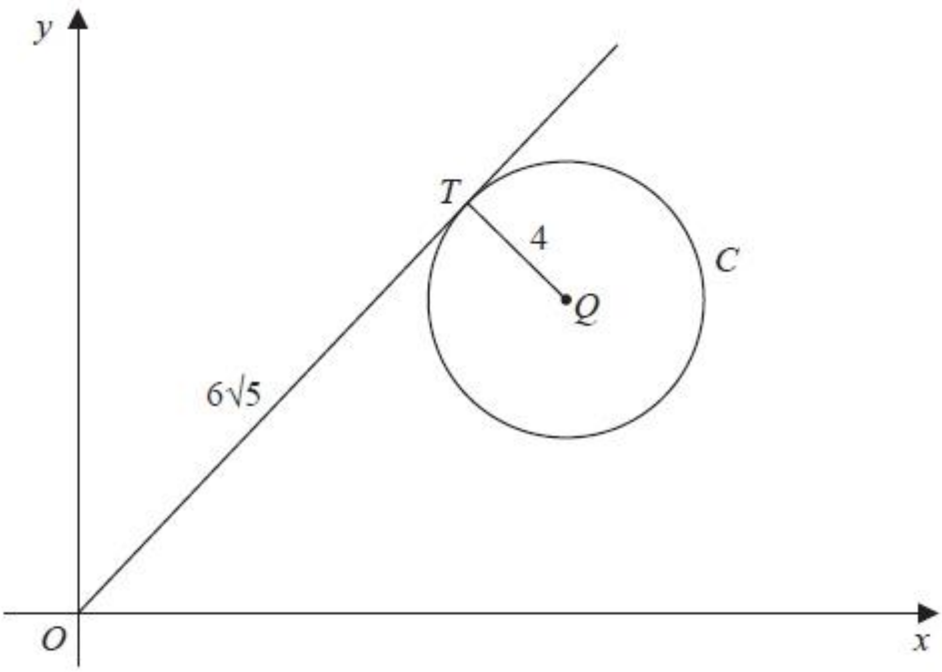


Figure 3

Figure 3 shows a circle *C* with centre *Q* and radius 4 and the point *T* which lies on *C*.

The tangent to *C* at the point *T* passes through the origin *O* and $OT = 6\sqrt{5}$

Given that the coordinates of *Q* are (11, *k*), where *k* is a positive constant,

(a) find the exact value of *k*,

(3)

(b) find an equation for *C*.

(2)

Q13.

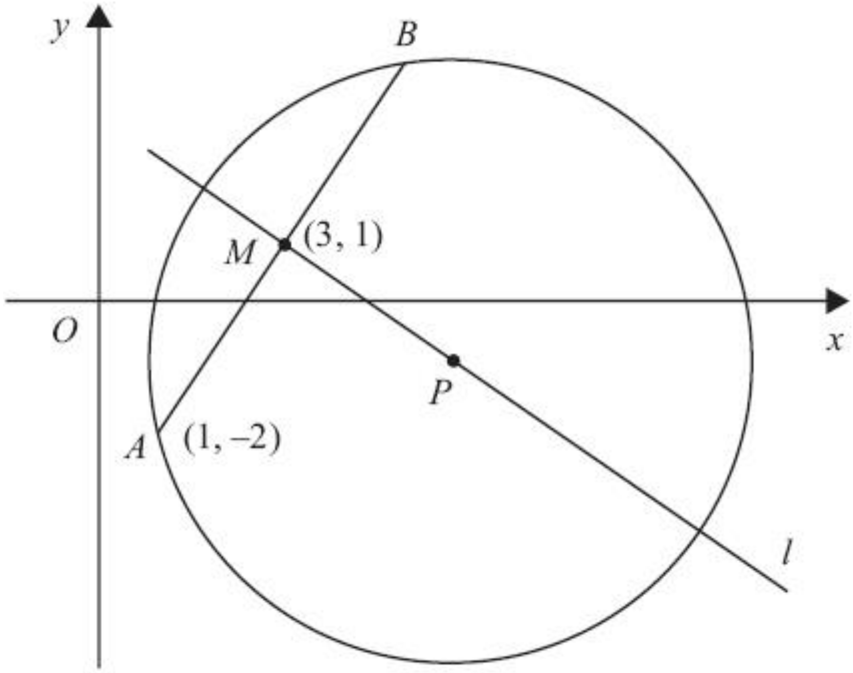


Figure 3

The points A and B lie on a circle with centre P, as shown in Figure 3.
The point A has coordinates $(1, -2)$ and the mid-point M of AB has coordinates $(3, 1)$.
The line l passes through the points M and P.

(a) Find an equation for l.

(4)

Given that the x-coordinate of P is 6,

(b) use your answer to part (a) to show that the y-coordinate of P is -1,

(1)

(c) find an equation for the circle.

(4)

Q14.

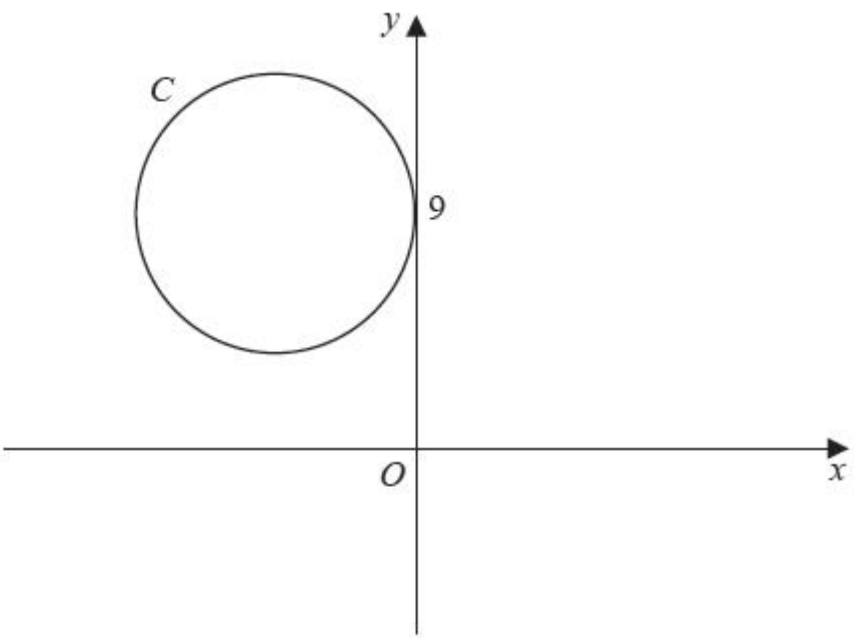


Figure 4

The circle C has radius 5 and touches the y-axis at the point (0, 9), as shown in Figure 4.

(a) Write down an equation for the circle C, that is shown in Figure 4.

(3)

A line through the point $P(8, -7)$ is a tangent to the circle C at the point T.

(b) (b) Find the length of PT.

(3)

Q15.

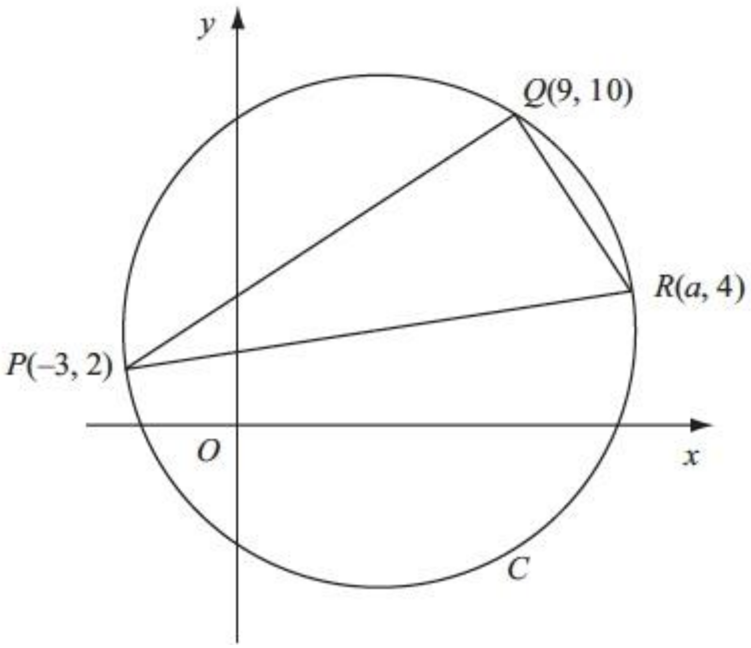


Figure 2

The points P(−3, 2), $Q(9, 10)$ and $R(a, 4)$ lie on the circle C, as shown in Figure 2. Given that PR is a diameter of C,

(a) show that $a = 13$,

(3)

(b) find an equation for C.

(5)