Edexcel New GCE A Level Maths workbook Binomial Expansion

Edited by: K V Kumaran

Binomial expansions

Binomial expansion of $(1 + x)^n$ for positive integer n.

Expansion of $(a + bx)^n$ may be required.

The notations $n!$ and $\binom{n}{r}$.

Pascal's Triangle

Pascal's Triangle looks like this

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

When we expand expression such as $(x + y)^4$ we find it follows the same pattern.

Example 1. $(x + y)^4 = x^2 + 2xy + y^2$ Notice the pattern of the coefficients (1,2,1)

Example 2. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y$ Here the coefficients are (1,3,3,1)

The key is to notice the 3 patterns:-

1. The coefficients follow Pascal's triangle sequence
2. The powers of one decrease while the other increase
3. The total power in each expression adds up to the original power in the questions

Example 3. $(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^3y^2 + 5xy^4 + y^5$

So now we know this:

$(a + b)^0 = 1$ *Line* 1

$(a + b)^1 = 1a + 1b$ *Line* 2

$(a + b)^2 = 1a^2 + 2ab + 1b^2$ *Line* 3

$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$ *Line* 4

$(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$ *Line* 5

Example 4. Expand $(2x + 3y)^3$ using Pascal's triangle

4th line is 1,3,3,1

$= 1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3$

$= 8x^3 + (3 \times 4x^2 \times 3y) + (3 \times 2x \times 9y^2) + (1 \times 27y^3)$

$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$

Example 5. Fully expand $(1 + 3x)(1 + 2x)^3$

$$(1 + 2x)^3 = 1(1)^3(2x)^0 + 3(1)^2(2x)^1 + 3(1)^1(2x)^2 + 1(1)^0(2x)^3$$

$$= 1 + 6x + 12x^2 + 8x^3$$

$$(1 + 3x)(1 + 2x)^3 = (1 + 3x)(1 + 6x + 12x^2 + 8x^3)$$

$$= 1 + 6x + 12x^2 + 8x^3 + 3x + 18x^2 + 36x^3 + 24x^4$$

$$= 1 + 9x + 30x^2 + 44x^3 + 24x^4$$

Factorial Notation

A factorial notation looks like this $n! = n \times (n-1) \times (n-2) \ldots\ldots$

Example 1. $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

To find the number of ways of choosing r items from a group of n items is written as

$${}^{n}C_{r} \text{ or } \binom{n}{r}$$

This is calculated by $\dfrac{n!}{r!(n-r)!}$

Example 2. Find ${}^{5}C_{3}$

$$= \frac{5!}{2! \times 3!}$$

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 4}$$

$$= \frac{120}{2 \times 6}$$

$$= 10$$

Example 3. Find $\binom{7}{4}$

$$= \frac{7!}{3! \times 4!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1}$$

$$= 35$$

Example 4. 4 People need to sit down but 2 want to sit together. How many different combinations are there?

Using ${}^{n}C_{r} = {}^{4}C_{2}$

$$= \frac{4!}{2!2!}$$

$$= \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1}$$

$$= 6$$

The Connection between Combination Notation and Pascal's Triangle

Look at this:-

$^4C_0 = 1 \quad ^4C_1 = 4 \quad ^4C_2 = 6 \quad ^4C_3 = 4 \quad ^4C_4 = 1$

These values are the same as the Index 4 line

$\therefore$ 6th line would be $\binom{6}{0} \; \binom{6}{1} \; \binom{6}{2} \; \binom{6}{3} \; \binom{6}{4} \; \binom{6}{5} \; \binom{6}{6}$

Using Factorial Notation to work out the Coefficients in the Binomial Expansion

The Binomial Expansion is:-

$$(a+b)^n = (a+b)(a+b)(a+b)\ldots\ldots..(a+b)$$

$$= ({}^nC_0)a^n b^0 + ({}^nC_1)a^{n-1}b^1 + ({}^nC_2)a^{n-2}b^2 + \ldots\ldots\ldots..({}^nC_n)a^{n-n}b^n$$

or $$\binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2\ldots\ldots\ldots..\binom{n}{n}a^{n-n}b^n$$

Example 1. Use the binomial theorem to find the expansion of $(3-2x)^5$

$$= \binom{5}{0}3^5(-2x)^0 + \binom{5}{1}3^4(-2x)^1 + \binom{5}{2}3^3(-2x)^2 + \binom{5}{3}3^2(-2x)^3$$

$$+ \binom{5}{4}3^1(-2x)^4 + \binom{5}{5}3^0(-2x)^5$$

$$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$$

Example 2. a) Write down the first 4 terms of the expansion $(1-\frac{x}{10})^6$

$$= \binom{6}{0}(1)^6\left(-\frac{x}{10}\right)^0 + \binom{6}{1}(1)^5\left(-\frac{x}{10}\right)^1 + \binom{6}{2}(1)^4\left(-\frac{x}{10}\right)^2 + \binom{6}{3}(1)^3\left(-\frac{x}{10}\right)^3$$

$$= 1 + \left(6 \times \frac{x}{10}\right) + \left(15 \times \frac{x^2}{100}\right) + \left(20 \times -\frac{x^3}{1000}\right)$$

$$= 1 - \frac{6x}{10} + \frac{15x^2}{100} - \frac{20x^3}{1000}$$

$$= 1 - 0{\cdot}6x + 0{\cdot}15x^2 - 0{\cdot}02x^3$$

b) By substituting an appropriate value of x, find an approximate value to $(0.99)^6$. Use your calculator to find the degree of accuracy of your approximation.

This means we want $\left(1 - \frac{x}{10}\right) = 0{\cdot}99$

$$0{\cdot}01 = \frac{x}{10}$$

$$0{\cdot}1 = x$$

substitute x = 0.1 into $(1 - 0{\cdot}6x + 0{\cdot}15x^2 - 0{\cdot}02x^3)$

$= 1 - (0{\cdot}6 \times 0{\cdot}1) + (0{\cdot}15 \times 0{\cdot}1^2) - (0{\cdot}02 \times 0{\cdot}1^3)$

$= 1 - 0{\cdot}06 + 0{\cdot}0015 - 0{\cdot}00002$

$= 0{\cdot}94148$

Using a calculator $0{\cdot}99^6 = 0{\cdot}941480149$

so approximation is accurate to 5dp as this how far the two answers are the same

Homework Questions 1 – Using Pascal's Triangle to Expand Brackets

1. Complete Pascal's triangle for the first 6 rows

2. Use pascal's triangle to find the expansions of the following
   a) $(x + y)^3$

   b) $(a - b)^4$

   c) $(1 + x)^5$

3. Find the coefficient of x^2 in the following expansions
   a) $(2 - x)^5$

   b) $(3 + 2x)^4$

   c) $(2 - 2x)^3$

4. The coefficient of x^2 in the following expansion is 24. Find the value of y

   $(x + 2y)^3$

5. Fully expand the following

   $(2 + x)(1 + x)^4$

Homework Questions 2 – Factorial and Combination Notation

1. Find the values of these factorial notations, show your working out

a) $7!$

c) $\frac{5!}{3!}$

b) $4!$

d) $\frac{8!}{5!}$

2. Find the values of these combination notations, show your working out

a) ${}^{4}C_{0}$

c) ${}^{5}C_{3}$

b) ${}^{7}C_{2}$

d) ${}^{6}C_{3}$

3. Find the value of these combination notations, show your working out

a) $\binom{3}{1}$

c) $\binom{6}{4}$

b) $\binom{7}{5}$

d) $\binom{5}{3}$

4. Prove that ${}^{6}C_{4} = \binom{6}{2}$

5. Write using combination notation Line 4 of Pascal's Triangle

Homework Questions 3 – Using Factorials in the Binomial Expansion

1. Write down how many terms there will be in the following expansions

a) $(x + 3)^5$

b) $(2 - x)^7$

c) $(x + 3y)^4$

d) $(3 - 2x)^8$

2. Write down the full expansions of the following, show all your working out

a) $(x + 4)^4$

b) $(2x - 3y)^3$

3. Find the first 3 terms of the expansions, show your working out

a) $(x + 2y)^3$

b) $(3x - y)^5$

4. Find the coefficient of x^2 of the following expansions

a) $(1 - x)^5$

b) $(3 - 4x)^4$

5. The coefficient of x^2 in the expansion below is 1944. Find the value of a

$$(3 + 2ax)^4$$

Homework Questions 4 – Using $(1+x)^n$ for Binomial Expansion

1. Find the first 4 terms of the following expansions using the binomial expansions

a) $(3 + x)^5$

b) $(2 - x)^7$

2. If x is so small that terms of x^3 and higher can be ignored, expand the following
$(2x + 3)(1 + 2x)^5$

1. Find the first three terms, in ascending powers of x, of the binomial expansion of $(3 + 2x)^5$, giving each term in its simplest form.

2. Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(3 - 2x)^5$, giving each term in its simplest form.

3. (*a*) Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1 + px)^{12}$, where p is a non-zero constant.

Given that, in the expansion of $(1 + px)^{12}$, the coefficient of x is $(-q)$ and the coefficient of x^2 is $11q$,

(*b*) find the value of p and the value of q.

4. (*a*) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$(1 + px)^9,$$

where p is a constant.

The first 3 terms are 1, $36x$ and qx^2, where q is a constant.

(*b*) Find the value of p and the value of q.

5. (*a*) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1 - 2x)^5$. Give each term in its simplest form.

(*b*) If x is small, so that x^2 and higher powers can be ignored, show that

$$(1 + x)(1 - 2x)^5 \approx 1 - 9x.$$

6. (*a*) Find the first four terms, in ascending powers of x, in the bionomial expansion of $(1 + kx)^6$, where k is a non-zero constant.

Given that, in this expansion, the coefficients of x and x^2 are equal, find

(*b*) the value of k,

(*c*) the coefficient of x^3.

7. (*a*) Find the first 4 terms of the expansion of $\left(1+\frac{x}{2}\right)^{10}$ in ascending powers of x, giving each term in its simplest form.

(*b*) Use your expansion to estimate the value of $(1.005)^{10}$, giving your answer to 5 decimal places.

8. (*a*) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1 + ax)^{10}$, where a is a non-zero constant. Give each term in its simplest form.

Given that, in this expansion, the coefficient of x^3 is double the coefficient of x^2,

(*b*) find the value of a.

9. (*a*) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$(2 + kx)^7$$

where k is a constant. Give each term in its simplest form.

Given that the coefficient of x^2 is 6 times the coefficient of x,

(*b*) find the value of k.

10. (*a*) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1+ ax)^7$, where a is a constant. Give each term in its simplest form.

Given that the coefficient of x^2 in this expansion is 525,

(*b*) find the possible values of a.

11. Given that $\binom{40}{4} = \dfrac{40!}{4!b!}$,

(*a*) write down the value of *b*.

In the binomial expansion of $(1 + x)^{40}$, the coefficients of x^4 and x^5 are *p* and *q* respectively.

(*b*) Find the value of $\dfrac{q}{p}$.

12. (*a*) Find the first 4 terms of the binomial expansion, in ascending powers of *x*, of

$$\left(1+\frac{x}{4}\right)^8,$$

giving each term in its simplest form.

(*b*) Use your expansion to estimate the value of $(1.025)^8$, giving your answer to 4 decimal places.

13. (*a*) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$\left(2 - \frac{x}{8}\right)^{10}$$

giving each term in its simplest form.

$$f(x) = \left(2 - \frac{x}{8}\right)^{10}(a + bx), \text{ where } a \text{ and } b \text{ are constants}$$

Given that the first two terms, in ascending powers of x in the series expansion of f(x), are 256 and $352x$,

(*b*) find the value of a,

(*c*) find the value of b.

14. The first 3 terms, in ascending powers of x, in the binomial expansion of $(1 + ax)^{20}$ are given by

$$1 + 4x + px^2$$

where a and p are constants.

(*a*) Find the value of a.

(*b*) Find the value of p.

One of the terms in the binomial expansion of $(1 + ax)^{20}$ is qx^4, where q is a constant.

(*c*) Find the value of q.