OCR Core Maths 2

Past paper questions Differentiation

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Differentiation

- We now turn to calculus[3]. Calculus ≡ Differentiation + Integration. You will discover integration in C2.
- Differentiation allows us to calculate the 'gradient function' $\frac{dy}{dx}$. This tells us how the gradient on the original function y changes with x. $\frac{dy}{dx}$ *is* the gradient of a curve. So if you need to find where on a curve the gradient is 7, then you solve $\frac{dy}{dx} = 7$.
- Two alternative notations for derivatives are $\frac{dy}{dx} \equiv f'(x) \equiv y'$.
- The "rules" are;

$$\begin{aligned}
y &= \text{constant} && \Rightarrow && \frac{dy}{dx} = 0, \\
y &= ax && \Rightarrow && \frac{dy}{dx} = a, \\
y &= ax^n && \Rightarrow && \frac{dy}{dx} = anx^{n-1}.
\end{aligned}$$

- For example:

$$\begin{aligned}
y = 4x^4 - 3x^2 + 2x - 5 \quad &\Rightarrow \quad \frac{dy}{dx} = 16x^3 - 6x + 2, \\
y = 4x^{\frac{5}{4}} + 3x^{\frac{4}{5}} \quad &\Rightarrow \quad \frac{dy}{dx} = 5x^{\frac{1}{4}} + \frac{12}{5}x^{-\frac{1}{5}}.
\end{aligned}$$

- You must expand brackets or carry out divisions *before* you differentiate[4]. For example:

$$\begin{aligned}
y = x^2(x-3)^2 \quad &\Rightarrow \quad y = x^4 - 6x^3 + 9x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 18x, \\
y = \frac{x^7 + x}{x^6} \quad &\Rightarrow \quad y = x + x^{-5} \quad \Rightarrow \quad \frac{dy}{dx} = 1 - 5x^{-6} = 1 - \frac{5}{x^6}.
\end{aligned}$$

- We can use differentiation to find the equation of tangents and normals to curves at specified points. For example find the equation of the normal to the curve $y = x^3 + 2x^2 - 5x - 1$ when $x = 1$.

  Firstly we need the y-coordinate: $x = 1 \quad \Rightarrow \quad y = -3$.

  Secondly $\frac{dy}{dx} = 3x^2 + 4x - 5$. Into this we put $x = 1$, so $\frac{dy}{dx} = 2$. Therefore the *normal* has gradient $-\frac{1}{2}$. So

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y + 3 &= -\tfrac{1}{2}(x - 1) \\
x + 2y + 5 &= 0.
\end{aligned}$$

- If asked to show that $y + 5x + 17 = 0$ is tangent to the curve $y = x^2 + 3x - 1$, there are two methods to do this:

  1. Find where $y = x^2 + 3x - 1$ and $y + 5x + 17 = 0$ cross. Solving simultaneously we gain the quadratic $-5x - 17 = x^2 + 3x - 1$ which simplifies and factorises to $(x + 4)(x + 4) = 0$. This gives a *repeated* root, so the line intersects the curve once and we can therefore conclude that the line *must* be a tangent. [I prefer this method.]
  2. The line $y + 5x + 17 = 0$ has gradient -5. Therefore we need to find where on $y = x^2 + 3x - 1$ the gradient is -5. Therefore we differentiate $y = x^2 + 3x - 1$ to get $\frac{dy}{dx} = 2x + 3$ and put $\frac{dy}{dx} = -5$. This gives $x = -4$. On the curve, when $x = -4$, $y = 3$, so to find the equation of the tangent

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -5(x + 4)$$
$$x + 5y + 17 = 0, \text{ as required.}$$

- Another example: Given that the curve $y = ax^3 + 4x^2 + bx + 1$ passes through the point $(-1, 5)$ and (at that point) the tangent is parallel to the line $y + 4x + 1 = 0$. Find a and b.

  There's quite a bit going on here, so take it a bit at a time. Since the curve passes through $(-1, 5)$, then $x = -1$ and $y = 5$ must be a solution to the curve's equation, so $5 = -a + 4 - b + 1$ which simplifies to $a + b = 0$. The line given has gradient -4, so we need to set $\frac{dy}{dx} = -4$ when $x = -1$. So $\frac{dy}{dx} = 3ax^2 + 8x + b$ which gives $-4 = 3a - 8 + b$. These solve to $a = 2$, $b = -2$.

Investigating Shapes of Graphs

- Stationary points are where the gradient of curve is zero. They are either maxima, minima or points of inflection. To find the turning points of a curve we must find $\frac{dy}{dx}$ and then set $\frac{dy}{dx} = 0$ and solve for x.
- To determine the nature of a turning point we can consider the sign of the gradient either side of the turning point. Present this in a table. In the example of $y = x^2 + 2x + 3$ we find $\frac{dy}{dx} = 2x + 2$ so we solve $0 = 2x + 2$ to give the turning point when $x = -1$:

x	$x < -1$	-1	$x > -1$
dy/dx	negative	0	positive
		minimum	

- We can also use the second derivative to determine the nature of a turning point. This is found by differentiating the function twice;

$$y = 2x^3 + 3x^2 - 2x + 4 \quad \Rightarrow \quad \frac{dy}{dx} = 6x^2 + 6x - 2 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 12x + 6.$$

You then evaluate the second derivative with the x value at the turning point and look at its sign. If it is positive it is a minimum, if it is negative it is a maximum. If it is zero then it is *probably* a point of inflection, but you need to do the above analysis either side of the turning point.

- For example, determine the nature of the stationary points on $y = 4x^3 - 21x^2 + 18x + 3$. So

$$y = 4x^3 - 21x^2 + 18x + 3 \quad \Rightarrow \quad \frac{dy}{dx} = 12x^2 - 42x + 18 = 0 \quad \Rightarrow \quad x = 3 \text{ or } x = \tfrac{1}{2}.$$

Therefore the stationary points are $(3, -24)$ and $(\frac{1}{2}, \frac{29}{4})$. We therefore need the second derivative and evaluate it at 3 and $\frac{1}{2}$.

$$\begin{aligned} \frac{d^2y}{dx^2} &= 24x - 42 \\ \left.\frac{d^2y}{dx^2}\right|_{x=3} &= 24 \times 3 - 42 = 30 > 0 \text{ therefore } (3, -24) \text{ is a minimum.} \\ \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} &= 24 \times \tfrac{1}{2} - 42 = -30 < 0 \text{ therefore } (\tfrac{1}{2}, \tfrac{29}{4}) \text{ is a maximum.} \end{aligned}$$

- A function will be *increasing* when $\frac{dy}{dx}$ is positive and *decreasing* when $\frac{dy}{dx}$ is negative[5]. This is obvious if you consider a sketch.

For example, find the set of values of x for which $y = -2x^3 + 3x^2 + 12x + 1$ is decreasing. First differentiate and recognise we want $\frac{dy}{dx}$ to be negative.

$$\begin{aligned} y &= -2x^3 + 3x^2 + 12x + 1 \\ \frac{dy}{dx} &= -6x^2 + 6x + 12 \\ 0 &> -6x^2 + 6x + 12 \qquad \left(\text{note that we place } \frac{dy}{dx} < 0\right) \\ x^2 - x - 2 &> 0 \\ (x - 2)(x + 1) &> 0. \end{aligned}$$

This quadratic inequality solves to $x < -1$ or $x > 2$ which are the values of x for which the curve is decreasing.

Applications of Differentiation

- Differentiation can be used to work out "rates of change". In GCSE Physics you learnt that acceleration is the rate of change of velocity; you will then have learnt that $a = \frac{v-u}{t}$. However, at a higher level rates of change are calculated by differentiating with respect to time. So we now view acceleration as $a = \frac{dv}{dt}$.

- As we have already seen, differentiation allows us to calculate the stationary point of a curve $y = f(x)$. We do this by calculating $\frac{dy}{dx}$ and setting it equal to zero. We can use this to help is in practical problems where we might want to maximise a quantity (e.g. profit) or to minimise a quantity (e.g. cost).

- Worked example: An open topped cuboidal box is to be made from a rectangular piece of metal 10cm by 16cm. Squares are to be cut from each corner and then the four flaps are to be folded up. Find the maximum volume attainable for the box and prove that it is a maximum.

1. Let x be the side length of the squares cut away, where $0 < x < 5$.
2. The volume of the box would therefore be

   $$V = x(16 - 2x)(10 - 2x) = 160x - 52x^2 + 4x^3.$$

   We imagine a graph of V against x and hope that there is a stationary point in the range $0 < x < 5$.
3. Differentiate V with respect to x and set equal to zero to find the stationary point;

   $$\frac{dV}{dx} = 160 - 104x + 12x^2 = 0 \quad \Rightarrow \quad x = 2 \text{ or } x = \frac{20}{3}.$$

4. Notice that x can't be $\frac{20}{3}$ because it is outside the range $0 < x < 5$. So we only consider $x = 2$.
5. If $x = 2$ then $V = 2 \times 12 \times 6 = 144$.
6. To demonstrate that $x = 2$ is a maximum we need the second derivative and evaluate it at $x = 2$.

   $$\frac{d^2V}{dx^2} = -104 + 24x = -104 + 24 \times 2 = -56 < 0 \text{ therefore a maximum.}$$

1.

Given that $f(x) = (x+1)^2(3x-4)$,

(i) express $f(x)$ in the form $ax^3 + bx^2 + cx + d$, [3]

(ii) find $f'(x)$, [2]

(iii) find $f''(x)$. [2]

Q6 June 2005

2.

(i) Given that $y = \frac{1}{3}x^3 - 9x$, find $\frac{dy}{dx}$. [2]

(ii) Find the coordinates of the stationary points on the curve $y = \frac{1}{3}x^3 - 9x$. [3]

(iii) Determine whether each stationary point is a maximum point or a minimum point. [3]

(iv) Given that $24x + 3y + 2 = 0$ is the equation of the tangent to the curve at the point (p, q), find p and q. [5]

Q10 June 2005

3.

Given that $y = 3x^5 - \sqrt{x} + 15$, find

(i) $\frac{dy}{dx}$, [3]

(ii) $\frac{d^2y}{dx^2}$. [2]

Q3 Jan 2006

4.

(i) Find the coordinates of the stationary points on the curve $y = x^3 - 3x^2 + 4$. [6]

(ii) Determine whether each stationary point is a maximum point or a minimum point. [3]

(iii) For what values of x does $x^3 - 3x^2 + 4$ increase as x increases? [2]

Q6 Jan 2006

5.

The points A (1, 3) and B (4, 21) lie on the curve $y = x^2 + x + 1$.

(i) Find the gradient of the line AB. [2]

(ii) Find the gradient of the curve $y = x^2 + x + 1$ at the point where $x = 3$. [2]

Q1 June 2006

6.

(i) Solve the equation $x^4 - 10x^2 + 25 = 0$. [4]

(ii) Given that $y = \frac{2}{5}x^5 - \frac{20}{3}x^3 + 50x + 3$, find $\frac{dy}{dx}$. [2]

(iii) Hence find the number of stationary points on the curve $y = \frac{2}{5}x^5 - \frac{20}{3}x^3 + 50x + 3$. [2]

Q6 June 2006

7.

A cuboid has a volume of $8\,\text{m}^3$. The base of the cuboid is square with sides of length x metres. The surface area of the cuboid is $A\,\text{m}^2$.

(i) Show that $A = 2x^2 + \frac{32}{x}$. [3]

(ii) Find $\frac{dA}{dx}$. [3]

(iii) Find the value of x which gives the smallest surface area of the cuboid, justifying your answer. [4]

Q8 June 2006

8.

Find $\frac{dy}{dx}$ in each of the following cases.

(i) $y = 5x + 3$ [1]

(ii) $y = \frac{2}{x^2}$ [3]

(iii) $y = (2x + 1)(5x - 7)$ [4]

Q7 Jan 2007

9.

(i) Find the coordinates of the stationary points of the curve $y = 27 + 9x - 3x^2 - x^3$. [6]

(ii) Determine, in each case, whether the stationary point is a maximum or minimum point. [3]

(iii) Hence state the set of values of x for which $27 + 9x - 3x^2 - x^3$ is an increasing function. [2]

Q8 Jan 2007

10.

The diagram shows a rectangular enclosure, with a wall forming one side. A rope, of length 20 metres, is used to form the remaining three sides. The width of the enclosure is x metres.

(i) Show that the enclosed area, A m^2, is given by

$$A = 20x - 2x^2.$$ [2]

(ii) Use differentiation to find the maximum value of A. [4]

Q5 June 2007

11.

(a) Given that $f(x) = x + \frac{3}{x}$, find $f'(x)$. [4]

(b) Find the gradient of the curve $y = x^{\frac{5}{2}}$ at the point where $x = 4$. [5]

Q7 June 2007

12.

(i) Find the coordinates of the stationary points on the curve $y = x^3 + x^2 - x + 3$. [6]

(ii) Determine whether each stationary point is a maximum point or a minimum point. [3]

(iii) For what values of x does $x^3 + x^2 - x + 3$ decrease as x increases? [2]

Q8 Jan 2008

13.

Given that $f(x) = 8x^3 + \frac{1}{x^3}$,

(i) find $f''(x)$, [5]

(ii) solve the equation $f(x) = -9$. [5]

Q10 Jan 2008

14.

Find the gradient of the curve $y = 8\sqrt{x} + x$ at the point whose x-coordinate is 9. [5]

Q5 June 2008

15.

The curve $y = x^3 - kx^2 + x - 3$ has two stationary points.

(i) Find $\frac{dy}{dx}$. [2]

(ii) Given that there is a stationary point when $x = 1$, find the value of k. [3]

(iii) Determine whether this stationary point is a minimum or maximum point. [2]

(iv) Find the x-coordinate of the other stationary point. [3]

Q8 June 2008

16.

Find $\frac{dy}{dx}$ in each of the following cases:

(i) $y = 10x^{-5}$, **[2]**

(ii) $y = \sqrt[4]{x}$, **[3]**

(iii) $y = x(x+3)(1-5x)$. **[4]**

Q5 Jan 2009

17.

The curve $y = x^3 + px^2 + 2$ has a stationary point when $x = 4$. Find the value of the constant p and determine whether the stationary point is a maximum or minimum point. **[7]**

Q9 Jan 2009

18.

A curve has equation $y = x^2 + x$.

(i) Find the gradient of the curve at the point for which $x = 2$. **[2]**

(ii) Find the equation of the normal to the curve at the point for which $x = 2$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. **[4]**

(iii) Find the values of k for which the line $y = kx - 4$ is a tangent to the curve. **[6]**

Q10 Jan 2009

19.

Given that $y = x^5 + \frac{1}{x^2}$, find

(i) $\frac{dy}{dx}$, **[3]**

(ii) $\frac{d^2y}{dx^2}$. **[2]**

Q1 June 2009

20.

(i) Solve the equation $9x^2 + 18x - 7 = 0$. **[3]**

(ii) Find the coordinates of the stationary point on the curve $y = 9x^2 + 18x - 7$. **[4]**

(iii) Sketch the curve $y = 9x^2 + 18x - 7$, giving the coordinates of all intercepts with the axes. **[3]**

(iv) For what values of x does $9x^2 + 18x - 7$ increase as x increases? **[1]**

Q10 June 2009

21.

The point P on the curve $y = k\sqrt{x}$ has x-coordinate 4. The normal to the curve at P is parallel to the line $2x + 3y = 0$.

(i) Find the value of k. **[6]**

(ii) This normal meets the x-axis at the point Q. Calculate the area of the triangle OPQ, where O is the point $(0, 0)$. **[5]**

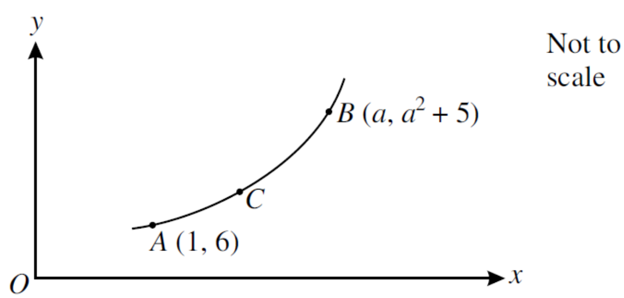
Q11 June 2009

22.

Find the equation of the normal to the curve $y = x^3 - 4x^2 + 7$ at the point $(2, -1)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. **[7]**

Q3 Jan 2010

23.



The diagram shows part of the curve $y = x^2 + 5$. The point A has coordinates $(1, 6)$. The point B has coordinates $(a, a^2 + 5)$, where a is a constant greater than 1. The point C is on the curve between A and B.

(i) Find by differentiation the value of the gradient of the curve at the point A. **[2]**

(ii) The line segment joining the points A and B has gradient 2.3. Find the value of a. **[4]**

(iii) State a possible value for the gradient of the line segment joining the points A and C. **[1]**

Q6 Jan 2010

24.

Given that $\mathrm{f}(x) = \frac{1}{x} - \sqrt{x} + 3$,

(i) find $\mathrm{f}'(x)$, **[3]**

(ii) find $\mathrm{f}''(4)$. **[5]**

Q9 Jan 2010

25.

Find the gradient of the curve $y = 2x + \frac{6}{\sqrt{x}}$ at the point where $x = 4$. **[5]**

Q6 June 2010

26.

(i) Find the coordinates of the stationary points of the curve $y = 2x^3 + 5x^2 - 4x$. **[6]**

(ii) State the set of values for x for which $2x^3 + 5x^2 - 4x$ is a decreasing function. **[2]**

(iii) Show that the equation of the tangent to the curve at the point where $x = \frac{1}{2}$ is $10x - 4y - 7 = 0$. **[4]**

(iv) Hence, with the aid of a sketch, show that the equation $2x^3 + 5x^2 - 4x = \frac{5}{2}x - \frac{7}{4}$ has two distinct real roots. **[2]**

Q10 June 2010

27.

Given that $y = \frac{5}{x^2} - \frac{1}{4x} + x$, find

(i) $\frac{\mathrm{d}y}{\mathrm{d}x}$, **[4]**

(ii) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. **[2]**

Q6 Jan 2011

28.

(i) Find the equation of the tangent to the curve $y = 7 + 6x - x^2$ at the point P where $x = 5$, giving your answer in the form $ax + by + c = 0$. **[6]**

(ii) This tangent meets the x-axis at Q. Find the coordinates of the mid-point of PQ. **[3]**

(iii) Find the equation of the line of symmetry of the curve $y = 7 + 6x - x^2$. **[2]**

(iv) State the set of values of x for which $7 + 6x - x^2$ is an increasing function. **[2]**

Q8 Jan 2011

29.

(i) Find the coordinates of the stationary point on the curve $y = 3x^2 - \frac{6}{x} - 2$. **[5]**

(ii) Determine whether the stationary point is a maximum point or a minimum point. **[2]**

Q8 June 2011

30.

A curve has equation $y = (2x - 1)(x + 3)(x - 1)$.

(i) Sketch the curve, indicating the coordinates of all points of intersection with the axes. **[3]**

(ii) Show that the gradient of the curve at the point $P(1, 0)$ is 4. **[6]**

(iii) The line l is parallel to the tangent to the curve at the point P. The curve meets l at the point where $x = -2$. Find the equation of l, giving your answer in the form $y = mx + c$. **[4]**

(iv) Determine whether l is a tangent to the curve at the point where $x = -2$. **[3]**

Q10 June 2011

31.

Given that $f(x) = \frac{4}{x} - 3x + 2$,

(i) find $f'(x)$, **[3]**

(ii) find $f''\left(\frac{1}{2}\right)$. **[4]**

Q6 Jan 2012

32.

A curve has equation $y = (x + 2)(x^2 - 3x + 5)$.

(i) Find the coordinates of the minimum point, justifying that it is a minimum. **[8]**

(ii) Calculate the discriminant of $x^2 - 3x + 5$. **[2]**

(iii) Explain why $(x + 2)(x^2 - 3x + 5)$ is always positive for $x > -2$. **[2]**

Q7 Jan 2012

33.

Find the equation of the normal to the curve $y = \frac{6}{x^2} - 5$ at the point on the curve where $x = 2$. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. **[7]**

Q6 June 2012

34.

(i) Find the coordinates of the stationary point on the curve $y = x^4 + 32x$. **[5]**

(ii) Determine whether this stationary point is a maximum or a minimum. **[2]**

(iii) For what values of x does $x^4 + 32x$ increase as x increases? **[1]**

Q8 June 2012

35.

(i) Solve the equation $x^2 - 6x - 2 = 0$, giving your answers in simplified surd form. **[3]**

(ii) Find the gradient of the curve $y = x^2 - 6x - 2$ at the point where $x = -5$. **[2]**

Q1 Jan 2013

36.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in each of the following cases:

(i) $y = \dfrac{(3x)^2 \times x^4}{x}$, **[3]**

(ii) $y = \sqrt[3]{x}$, **[3]**

(iii) $y = \dfrac{1}{2x^3}$. **[2]**

Q7 Jan 2013

37.

Find the coordinates of the points on the curve $y = \frac{1}{3}x^3 + \frac{9}{x}$ at which the tangent is parallel to the line $y = 8x + 3$. **[10]**

Q10 Jan 2013

38.

It is given that $\mathrm{f}(x) = \dfrac{6}{x^2} + 2x$.

(i) Find $\mathrm{f}'(x)$. **[3]**

(ii) Find $\mathrm{f}''(x)$. **[2]**

Q3 June 2013

39.

The curve $y = (1-x)(x^2+4x+k)$ has a stationary point when $x = -3$.

(i) Find the value of the constant k. **[7]**

(ii) Determine whether the stationary point is a maximum or minimum point. **[2]**

(iii) Given that $y = 9x - 9$ is the equation of the tangent to the curve at the point A, find the coordinates of A. **[5]**

Q10 June 2013

40.

Given that $y = 6x^3 + \frac{4}{\sqrt{x}} + 5x$, find

(i) $\frac{dy}{dx}$, **[4]**

(ii) $\frac{d^2y}{dx^2}$. **[2]**

Q6 June 2014

41.

A curve has equation $y = 3x^3 - 7x + \frac{2}{x}$.

(i) Verify that the curve has a stationary point when $x = 1$. **[5]**

(ii) Determine the nature of this stationary point. **[2]**

(iii) The tangent to the curve at this stationary point meets the y-axis at the point Q. Find the coordinates of Q. **[2]**

Q8 June 2014

42.

A curve has equation $y = (x+2)^2(2x-3)$.

(i) Sketch the curve, giving the coordinates of all points of intersection with the axes. **[3]**

(ii) Find an equation of the tangent to the curve at the point where $x = -1$. Give your answer in the form $ax + by + c = 0$. **[9]**

Q10 June 2014

43.

(a) Given that $f(x) = (x^2+3)(5-x)$, find $f'(x)$. **[4]**

(b) Find the gradient of the curve $y = x^{-\frac{1}{3}}$ at the point where $x = -8$. **[4]**

Q7 June 2015

44.

The curve $y = 2x^3 - ax^2 + 8x + 2$ passes through the point B where $x = 4$.

(i) Given that B is a stationary point of the curve, find the value of the constant a. **[5]**

(ii) Determine whether the stationary point B is a maximum point or a minimum point. **[2]**

(iii) Find the x-coordinate of the other stationary point of the curve. **[3]**

Q9 June 2015