OCR Core Maths 4

Past paper questions Differential equations

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Differential Equations

- If you are told that ("something") is proportional to ("something else") then we write ("something") $\propto$ ("something else"). This implies that

$$(\text{"something"}) = \pm k(\text{"something else"})$$

  for some *constant* k. k can then be determined by putting in one pair of values (x, y) into the equation. If you read that something is decreasing then use $-k$, if it is increasing then use $+k$. The expression "varies as" also implies proportionality between two quantities.

- The words "rate of change of (something)" $\Rightarrow \frac{d(\text{something})}{dt}$. Also the word "initially" $\Rightarrow$ $t = 0$. Also be on the lookout for phrases such as "where t is measured from now" implying $t = 0$ now.

- In simple cases you need to be able to construct a differential equation of a situation. For example: The number of people infected with bird flu (N) is growing at a rate proportional to the square of the number of people infected:

$$\frac{dN}{dt} \propto N^2 \quad \Rightarrow \quad \frac{dN}{dt} = +kN^2.$$

- The notation dy/dx lets us believe the it is a normal fraction. Although this is not the case we can manipulate it like a fraction in a differential equation. You must move the variables to different sides of the equation and integrate (separation of variables). Only add the ever-present "$+c$" to one side. For example solve

$$\frac{dy}{dx} = y^2 \cos x \qquad \Rightarrow \qquad \int \frac{1}{y^2}\,dy = \int \cos x\,dx \qquad \Rightarrow \qquad y = -\frac{1}{\sin x + c}.$$

$$\frac{dN}{dt} = +kN^2 \qquad \Rightarrow \qquad \int \frac{1}{N^2}\,dN = \int k\,dt \qquad \Rightarrow \qquad N = \frac{-1}{kt + c}.$$

  In the second example above you will notice that there are two constants; the constant of proportionality and the arbitrary integration constant. This means you will need to be given two pieces of data (t_1, N_1) and (t_2, N_2) to figure them both out.

- A final example involving partial fractions:

$$\begin{aligned}(3P+1)\frac{dP}{dt} &= kt(P-1)(P+3)\\ \int \tfrac{3P+1}{(P-1)(P+3)}\,dP &= \int kt\,dt\\ \int \tfrac{1}{P-1} + \tfrac{2}{P+3}\,dP &= \tfrac{kt^2}{2} + c\\ \ln(P-1) + 2\ln(P+3) &= \tfrac{kt^2}{2} + c\\ \ln(P-1)(P+3)^2 &= \tfrac{kt^2}{2} + c.\end{aligned}$$

- If the arbitrary constant is left unevaluated, then your solution represents the *general solution* of the differential equation. If you put a value in to work out its value then your solution is called the *particular solution*.

1.

Newton's law of cooling states that the rate at which the temperature of an object is falling at any instant is proportional to the difference between the temperature of the object and the temperature of its surroundings at that instant. A container of hot liquid is placed in a room which has a constant temperature of 20 °C. At time t minutes later, the temperature of the liquid is θ °C.

(i) Explain how the information above leads to the differential equation

$$\frac{d\theta}{dt} = -k(\theta - 20),$$

where k is a positive constant. [2]

(ii) The liquid is initially at a temperature of 100 °C. It takes 5 minutes for the liquid to cool from 100 °C to 68 °C. Show that

$$\theta = 20 + 80e^{-(\frac{1}{5}\ln\frac{5}{3})t}.$$ [8]

(iii) Calculate how much longer it takes for the liquid to cool by a further 32 °C. [3]

Q9 June 2005

2.

(i) Solve the differential equation

$$\frac{dy}{dx} = \frac{2-x}{y-3},$$

giving the particular solution that satisfies the condition $y = 4$ when $x = 5$. [5]

(ii) Show that this particular solution can be expressed in the form

$$(x-a)^2 + (y-b)^2 = k,$$

where the values of the constants a, b and k are to be stated. [3]

(iii) Hence sketch the graph of the particular solution, indicating clearly its main features. [3]

Q8 Jan 2006

3.

A forest is burning so that, t hours after the start of the fire, the area burnt is A hectares. It is given that, at any instant, the rate at which this area is increasing is proportional to A^2.

(i) Write down a differential equation which models this situation. [2]

(ii) After 1 hour, 1000 hectares have been burnt; after 2 hours, 2000 hectares have been burnt. Find after how many hours 3000 hectares have been burnt. [6]

Q5 June 2006

4.

(i) Find the general solution of the differential equation

$$\frac{\sec^2 y}{\cos^2(2x)} \frac{\mathrm{d}y}{\mathrm{d}x} = 2. \qquad [7]$$

(ii) For the particular solution in which $y = \frac{1}{4}\pi$ when $x = 0$, find the value of y when $x = \frac{1}{6}\pi$. [3]

Q9 Jan 2007

5.

The height, h metres, of a shrub t years after planting is given by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{6-h}{20}.$$

A shrub is planted when its height is 1 m.

(i) Show by integration that $t = 20\ln\left(\frac{5}{6-h}\right)$. [6]

(ii) How long after planting will the shrub reach a height of 2 m? [1]

(iii) Find the height of the shrub 10 years after planting. [2]

(iv) State the maximum possible height of the shrub. [1]

Q8 June 2007

6.

Water flows out of a tank through a hole in the bottom and, at time t minutes, the depth of water in the tank is x metres. At any instant, the rate at which the depth of water in the tank is decreasing is proportional to the square root of the depth of water in the tank.

(i) Write down a differential equation which models this situation. [2]

(ii) When $t = 0, x = 2$; when $t = 5, x = 1$. Find t when $x = 0.5$, giving your answer correct to 1 decimal place. [6]

Q8 Jan 2008

7.

(i) Show that, if $y = \operatorname{cosec} x$, then $\frac{dy}{dx}$ can be expressed as $-\operatorname{cosec} x \cot x$. [3]

(ii) Solve the differential equation

$$\frac{dx}{dt} = -\sin x \tan x \cot t,$$

given that $x = \frac{1}{6}\pi$ when $t = \frac{1}{2}\pi$. [5]

Q7 June 2008

8.

A liquid is being heated in an oven maintained at a constant temperature of 160 °C. It may be assumed that the rate of increase of the temperature of the liquid at any particular time t minutes is proportional to $160 - \theta$, where θ °C is the temperature of the liquid at that time.

(i) Write down a differential equation connecting θ and t. **[2]**

When the liquid was placed in the oven, its temperature was 20 °C and 5 minutes later its temperature had risen to 65 °C.

(ii) Find the temperature of the liquid, correct to the nearest degree, after another 5 minutes. **[9]**

Q9 Jan 2009

9.

A tank contains water which is heated by an electric water heater working under the action of a thermostat. The temperature of the water, θ °C, may be modelled as follows. When the water heater is first switched on, $\theta = 40$. The heater causes the temperature to increase at a rate k_1 °C per second, where k_1 is a constant, until $\theta = 60$. The heater then switches off.

(i) Write down, in terms of k_1, how long it takes for the temperature to increase from 40 °C to 60 °C. **[1]**

The temperature of the water then immediately starts to decrease at a variable rate $k_2(\theta - 20)$ °C per second, where k_2 is a constant, until $\theta = 40$.

(ii) Write down a differential equation to represent the situation as the temperature is decreasing. **[1]**

(iii) Find the total length of time for the temperature to increase from 40 °C to 60 °C and then decrease to 40 °C. Give your answer in terms of k_1 and k_2. **[8]**

Q9 June 2009

10.

(i) Express $\dfrac{1}{(3-x)(6-x)}$ in partial fractions. **[2]**

(ii) In a chemical reaction, the amount x grams of a substance at time t seconds is related to the rate at which x is changing by the equation

$$\frac{dx}{dt} = k(3-x)(6-x),$$

where k is a constant. When $t = 0$, $x = 0$ and when $t = 1$, $x = 1$.

(a) Show that $k = \frac{1}{3}\ln\frac{5}{4}$. **[7]**

(b) Find the value of x when $t = 2$. **[4]**

Q10 Jan 2010

11.

(i) Find the quotient and the remainder when $x^2 - 5x + 6$ is divided by $x - 1$. **[3]**

(ii) **(a)** Find the general solution of the differential equation

$$\left(\frac{x-1}{x^2-5x+6}\right)\frac{dy}{dx} = y - 5.$$ **[3]**

(b) Given that $y = 7$ when $x = 8$, find y when $x = 6$. **[4]**

Q8 June 2010

12.

Paraffin is stored in a tank with a horizontal base. At time t minutes, the depth of paraffin in the tank is x cm. When $t = 0$, $x = 72$. There is a tap in the side of the tank through which the paraffin can flow. When the tap is opened, the flow of the paraffin is modelled by the differential equation

$$\frac{dx}{dt} = -4(x-8)^{\frac{1}{3}}.$$

(i) How long does it take for the level of paraffin to fall from a depth of 72 cm to a depth of 35 cm? **[7]**

(ii) The tank is filled again to its original depth of 72 cm of paraffin and the tap is then opened. The paraffin flows out until it stops. How long does this take? **[3]**

Q9 Jan 2011

13.

The gradient of a curve at the point (x, y), where $x > -2$, is given by

$$\frac{dy}{dx} = \frac{1}{3y^2(x+2)}.$$

The points $(1, 2)$ and $(q, 1.5)$ lie on the curve. Find the value of q, giving your answer correct to 3 significant figures. **[7]**

Q7 June 2011

14.

(i) Write down the derivative of $\sqrt{y^2+1}$ with respect to y. **[1]**

(ii) Given that $\frac{dy}{dx} = \frac{(x-1)\sqrt{y^2+1}}{xy}$ and that $y = \sqrt{e^2 - 2e}$ when $x = e$,

find a relationship between x and y. **[8]**

Q10 Jan 2012

15.

Solve the differential equation

$$e^{2y}\frac{dy}{dx} + \tan x = 0,$$

given that $x = 0$ when $y = 0$. Give your answer in the form $y = f(x)$. **[6]**

Q4 June 2012

16.

The temperature of a freezer is –20 °C. A container of a liquid is placed in the freezer. The rate at which the temperature, θ °C, of a liquid decreases is proportional to the difference in temperature between the liquid and its surroundings. The situation is modelled by the differential equation

$$\frac{d\theta}{dt} = -k(\theta + 20),$$

where time t is in minutes and k is a positive constant.

(i) Express θ in terms of t, k and an arbitrary constant. **[3]**

Initially the temperature of the liquid in the container is 40 °C and, at this instant, the liquid is cooling at a rate of 3 °C per minute. The liquid freezes at 0 °C.

(ii) Find the value of k and find also the time it takes (to the nearest minute) for the liquid to freeze. **[5]**

The procedure is repeated on another occasion with a different liquid. The initial temperature of this liquid is 90 °C. After 19 minutes its temperature is 0 °C.

(iii) Without any further calculation, explain what you can deduce about the value of k in this case. **[1]**

Q9 Jan 2013

17.

At time t seconds, the radius of a spherical balloon is r cm. The balloon is being inflated so that the rate of increase of its radius is inversely proportional to the square root of its radius. When $t = 5$, $r = 9$ and, at this instant, the radius is increasing at 1.08 cm s^{-1}.

(i) Write down a differential equation to model this situation, and solve it to express r in terms of t. **[7]**

(ii) How much air is in the balloon initially? **[2]**

[The volume of a sphere is $V = \frac{4}{3}\pi r^3$.]

Q8 June 2013

18.

A container in the shape of an inverted cone of radius 3 metres and vertical height 4.5 metres is initially filled with liquid fertiliser. This fertiliser is released through a hole in the bottom of the container at a rate of 0.01 m^3 per second. At time t seconds the fertiliser remaining in the container forms an inverted cone of height h metres.

[The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.]

(i) Show that $h^2 \frac{dh}{dt} = -\frac{9}{400\pi}$. **[5]**

(ii) Express h in terms of t. **[4]**

(iii) Find the time it takes to empty the container, giving your answer to the nearest minute. **[2]**

Q10 June 2014

19.

In the year 2000 the population density, P, of a village was 100 people per km^2, and was increasing at the rate of 1 person per km^2 per year. The rate of increase of the population density is thought to be inversely proportional to the size of the population density. The time in years after the year 2000 is denoted by t.

(i) Write down a differential equation to model this situation, and solve it to express P in terms of t. **[6]**

(ii) In 2008 the population density of the village was 108 people per km^2 and in 2013 it was 128 people per km^2. Determine how well the model fits these figures. **[2]**

Q8 June 2015