Edexcel

Pure Mathematics

Year 2

Differential Equations

and

Connected Rate of change

Past paper questions from Core Maths 4 and IAL C34

Edited by: K V Kumaran

Past paper questions differential equations

1. Liquid is pouring into a container at a constant rate of 20 $cm^3\ s^{-1}$ and is leaking out at a rate proportional to the volume of the liquid already in the container.

(*a*) Explain why, at time t seconds, the volume, V cm^3, of liquid in the container satisfies the differential equation

$$\frac{dV}{dt} = 20 - kV,$$

where k is a positive constant. **(2)**

The container is initially empty.

(*b*) By solving the differential equation, show that

$$V = A + Be^{-kt},$$

giving the values of A and B in terms of k.

(6)

Given also that $\frac{dV}{dt} = 10$ when $t = 5$,

(*c*) find the volume of liquid in the container at 10 s after the start. **(5)**

(C4, June 2005 Q8)

2. The volume of a spherical balloon of radius r cm is V cm^3, where $V = \frac{4}{3}\pi r^3$.

(*a*) Find $\frac{dV}{dr}$. **(1)**

The volume of the balloon increases with time t seconds according to the formula

$$\frac{dV}{dt} = \frac{1000}{(2t+1)^2}, \quad t \geq 0.$$

(*b*) Using the chain rule, or otherwise, find an expression in terms of r and t for $\frac{dr}{dt}$.

(2)

(*c*) Given that $V = 0$ when $t = 0$, solve the differential equation $\frac{dV}{dt} = \frac{1000}{(2t+1)^2}$, to obtain V in terms of t. **(4)**

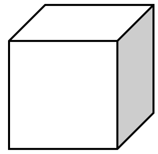
(*d*) Hence, at time $t = 5$,

(i) find the radius of the balloon, giving your answer to 3 significant figures, **(3)**

(ii) show that the rate of increase of the radius of the balloon is approximately 2.90×10^{-2} cm s^{-1}. **(2)**

(C4, Jan 2006 Q7)

3.



At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3.

The surface area of the cube is increasing at a constant rate of 8 cm^2 s^{-1}.

Show that

(*a*) $\dfrac{dx}{dt} = \dfrac{k}{x}$, where k is a constant to be found, **(4)**

(*b*) $\dfrac{dV}{dt} = 2V^{\frac{1}{3}}$. **(4)**

Given that $V = 8$ when $t = 0$,

(*c*) solve the differential equation in part (*b*), and find the value of t when $V = 16\sqrt{2}$.

(7)

(C4, June 2006 Q7)

4. (*a*) Express $\dfrac{2x-1}{(x-1)(2x-3)}$ in partial fractions. **(3)**

(*b*) Given that $x \geq 2$, find the general solution of the differential equation

$$(2x-3)(x-1)\frac{dy}{dx} = (2x-1)\,y.$$

(5)

(*c*) Hence find the particular solution of this differential equation that satisfies $y = 10$ at $x = 2$, giving your answer in the form $y = f(x)$.

(4)

(C4, Jan 2007 Q4)

5. A population growth is modelled by the differential equation

$$\frac{dP}{dt} = kP,$$

where P is the population, t is the time measured in days and k is a positive constant.

Given that the initial population is P_0,

(*a*) Solve the differential equation, giving P in terms of P_0, k and t.

(4)

Given also that $k = 2.5$,

(*b*) find the time taken, to the nearest minute, for the population to reach $2P_0$.

(3)

In an improved model the differential equation is given as

$$\frac{dP}{dt} = \lambda P \cos \lambda t,$$

where P is the population, t is the time measured in days and λ is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,

(*c*) solve the second differential equation, giving P in terms of P_0, λ and t. **(4)**

Given also that $\lambda = 2.5$,

(*d*) find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model.

(3)

(C4, June 2007 Q8)

6. Liquid is pouring into a large vertical circular cylinder at a constant rate of 1600 cm^3s^{-1} and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(*a*) Show that at time t seconds, the height h cm of liquid in the cylinder satisfies the differential equation

$$\frac{dh}{dt} = 0.4 - k\sqrt{h},$$

where k is a positive constant. **(3)**

When $h = 25$, water is leaking out of the hole at 400 cm^3s^{-1}.

(*b*) Show that $k = 0.02$.

(1)

(*c*) Separate the variables of the differential equation

$$\frac{dh}{dt} = 0.4 - 0.02\sqrt{h}$$

to show that the time taken to fill the cylinder from empty to a height of 100 cm is given by

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh .$$

(2)

Using the substitution $h = (20 - x)^2$, or otherwise,

(*d*) find the exact value of $\displaystyle\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh$. **(6)**

(*e*) Hence find the time taken to fill the cylinder from empty to a height of 100 cm, giving your answer in minutes and seconds to the nearest second.

(1)

(C4, Jan 2008 Q8)

7.

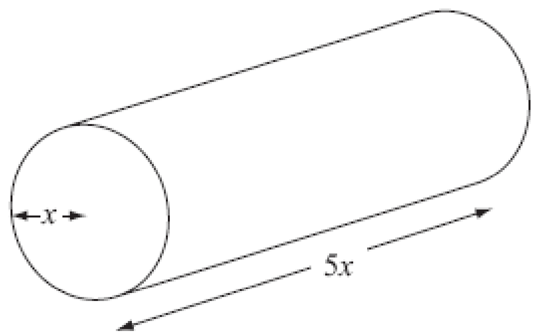


Figure 2

Figure 2 shows a right circular cylindrical metal rod which is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm.

The cross-sectional area of the rod is increasing at the constant rate of 0.032 $\text{cm}^2\,\text{s}^{-1}$.

(*a*) Find $\dfrac{dx}{dt}$ when the radius of the rod is 2 cm, giving your answer to 3 significant figures.

(4)

(*b*) Find the rate of increase of the volume of the rod when $x = 2$. **(4)**

(C4, June 2008 Q3)

8. (*a*) Express $\dfrac{2}{4 - y^2}$ in partial fractions. **(3)**

(*b*) Hence obtain the solution of

$$2 \cot x \frac{dy}{dx} = (4 - y^2)$$

for which $y = 0$ at $x = \dfrac{\pi}{3}$, giving your answer in the form $\sec^2 x = g(y)$.

(8)

(C4, June 2008 Q7)

9. (*a*) Find $\displaystyle\int \frac{9x + 6}{x}\, dx$, $x > 0$. **(2)**

(*b*) Given that $y = 8$ at $x = 1$, solve the differential equation

$$\frac{dy}{dx} = \frac{(9x + 6)y^{\frac{1}{3}}}{x}$$

giving your answer in the form $y^2 = g(x)$.

(6)

(C4, Jan 2010 Q5)

10.

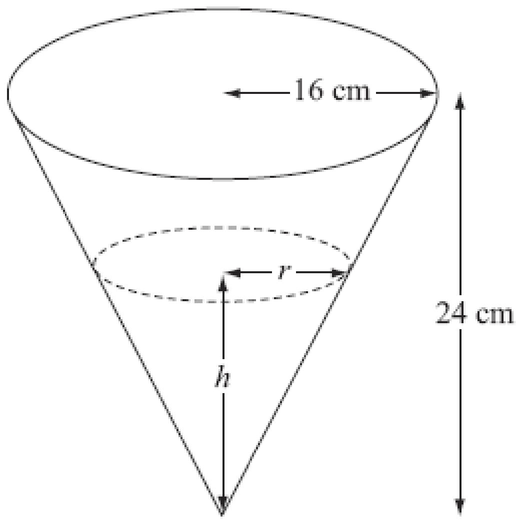


Figure 2

A container is made in the shape of a hollow inverted right circular cone. The height of the container is 24 cm and the radius is 16 cm, as shown in Figure 2. Water is flowing into the container. When the height of water is h cm, the surface of the water has radius r cm and the volume of water is V cm^3.

(*a*) Show that $V = \dfrac{4\pi h^3}{27}$. **(2)**

[*The volume V of a right circular cone with vertical height h and base radius r is given by the formula* $V = \dfrac{1}{3}\pi r^2 h$.

Water flows into the container at a rate of 8 cm^3 s^{-1}.

(*b*) Find, in terms of π, the rate of change of h when $h = 12$.

(5)

(C4, Jan 2009 Q5)

11. The area A of a circle is increasing at a constant rate of 1.5 cm^2 s^{-1}. Find, to 3 significant figures, the rate at which the radius r of the circle is increasing when the area of the circle is 2 cm^2.

(5)

(C4, Jan 2010 Q6)

12.

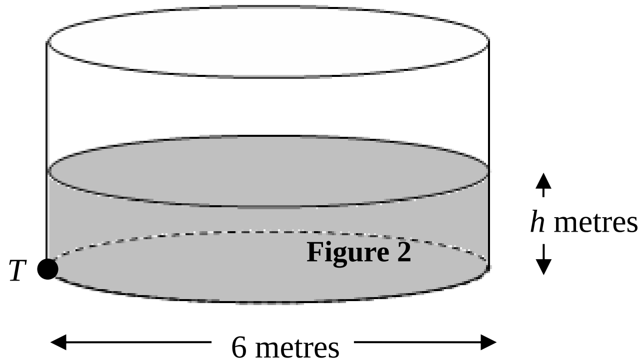


Figure 2 shows a cylindrical water tank. The diameter of a circular cross-section of the tank is 6 m. Water is flowing into the tank at a constant rate of $0.48\pi \text{ m}^3 \text{ min}^{-1}$. At time *t* minutes, the depth of the water in the tank is *h* metres. There is a tap at a point *T* at the bottom of the tank. When the tap is open, water leaves the tank at a rate of $0.6\pi h \text{ m}^3 \text{ min}^{-1}$.

(a) Show that *t* minutes after the tap has been opened

$$75\frac{\mathrm{d}h}{\mathrm{d}t} = (4 - 5h)$$ **(5)**

When $t = 0$, $h = 0.2$

(b) Find the value of *t* when $h = 0.5$ **(6)**

(C4, June 2010 Q8)

13. (*a*) Express $\dfrac{5}{(x-1)(3x+2)}$ in partial fractions. **(3)**

(*b*) Hence find $\displaystyle\int \frac{5}{(x-1)(3x+2)}\,\mathrm{d}x$, where $x > 1$. **(3)**

(*c*) Find the particular solution of the differential equation

$$(x-1)(3x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 5y, \quad x > 1,$$

for which $y = 8$ at $x = 2$. Give your answer in the form $y = \mathrm{f}(x)$. **(6)**

(C4, Jan 2011 Q3)

14.

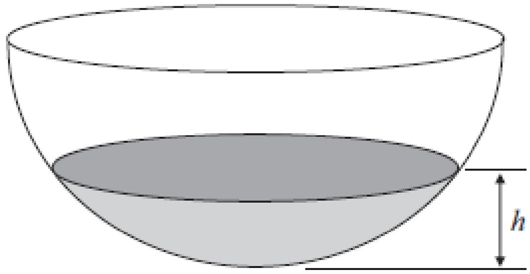


Figure 1

A hollow hemispherical bowl is shown in Figure 1. Water is flowing into the bowl.

When the depth of the water is h m, the volume V m^3 is given by

$$V = \frac{1}{12}\pi h^2(3 - 4h), \qquad 0 \leq h \leq 0.25.$$

(*a*) Find, in terms of π, $\frac{\mathrm{d}V}{\mathrm{d}h}$ when $h = 0.1$. **(4)**

Water flows into the bowl at a rate of $\frac{\pi}{800}$ m^3 s^{-1}.

(*b*) Find the rate of change of h, in m s^{-1}, when $h = 0.1$.

(2)

(C4, June 2011 Q3)

15. (*a*) Find $\int (4y+3)^{-\frac{1}{2}}\ \mathrm{d}y$.

(2)

(*b*) Given that y =1.5 at x = – 2, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{(4y+3)}}{x^2},$$

giving your answer in the form $y = \mathrm{f}(x)$.

(6)

(C4, June 2011 Q8)

16. (*a*) Express $\dfrac{1}{P(5-P)}$ in partial fractions.

(3)

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{15}P(5-P), \quad t \geq 0,$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that when $t = 0$, $P = 1$,

(*b*) solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b + c\mathrm{e}^{-\frac{1}{3}t}}$$

where a, b and c are integers.

(8)

(*c*) Hence show that the population cannot exceed 5000.

(1)

(C4, Jan 2012 Q8)

17.

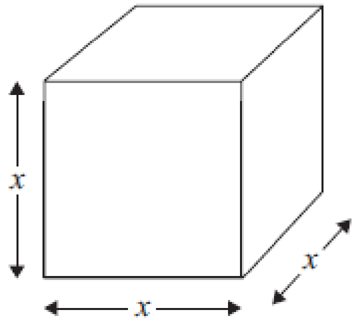


Figure 1

Figure 1 shows a metal cube which is expanding uniformly as it is heated.

At time t seconds, the length of each edge of the cube is x cm, and the volume of the cube is V cm^3.

(a) Show that $\dfrac{\mathrm{d}V}{\mathrm{d}x} = 3x^2$.

(1)

Given that the volume, V cm^3, increases at a constant rate of 0.048 cm^3 s^{-1},

(b) find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 8$,

(2)

(c) find the rate of increase of the total surface area of the cube, in cm^2 s^{-1}, when $x = 8$.

(3)

(C4, June 2012 Q2)

18. Given that $y = 2$ at $x = \dfrac{\pi}{4}$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{y\cos^2 x}.$$

(5)

(C4, June 2012 Q4)

19. A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at 3 °C and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is θ °C. The rate of change of the temperature of the water in the bottle is modelled by the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{(3-\theta)}{125}.$$

(*a*) By solving the differential equation, show that

$$\theta = A\mathrm{e}^{-0.008t} + 3,$$

where A is a constant.

(4)

Given that the temperature of the water in the bottle when it was put in the refrigerator was 16 °C,

(*b*) find the time taken for the temperature of the water in the bottle to fall to 10 °C, giving your answer to the nearest minute.

(5)

(C4, Jan 2013 Q8)

20. Water is being heated in a kettle. At time t seconds, the temperature of the water is θ°C.

The rate of increase of the temperature of the water at any time t is modelled by the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \lambda(120 - \theta), \qquad \theta \leq 100$$

where λ is a positive constant.

Given that $\theta = 20$ when $t = 0$,

(*a*) solve this differential equation to show that

$$\theta = 120 - 100\mathrm{e}^{-\lambda t}$$

(8)

When the temperature of the water reaches 100°C, the kettle switches off.

(*b*) Given that $\lambda = 0.01$, find the time, to the nearest second, when the kettle switches off.

(3)

(C4, June 2013, Q6)

21. In an experiment testing solid rocket fuel, some fuel is burned and the waste products are collected. Throughout the experiment the sum of the masses of the unburned fuel and waste products remains constant.

Let x be the mass of waste products, in kg, at time t minutes after the start of the experiment. It is known that at time t minutes, the rate of increase of the mass of waste products, in kg per minute, is k times the mass of unburned fuel remaining, where k is a positive constant.

The differential equation connecting x and t may be written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(M - x), \text{ where } M \text{ is a constant.}$$

(*a*) Explain, in the context of the problem, what $\frac{\mathrm{d}x}{\mathrm{d}t}$ and M represent.

(2)

Given that initially the mass of waste products is zero,

(*b*) solve the differential equation, expressing x in terms of k, M and t.

(6)

Given also that $x = \frac{1}{2}M$ when $t = \ln 4$,

(*c*) find the value of x when $t = \ln 9$, expressing x in terms of M, in its simplest form.

(4)

(C4, June 2013_R, Q8)

22.

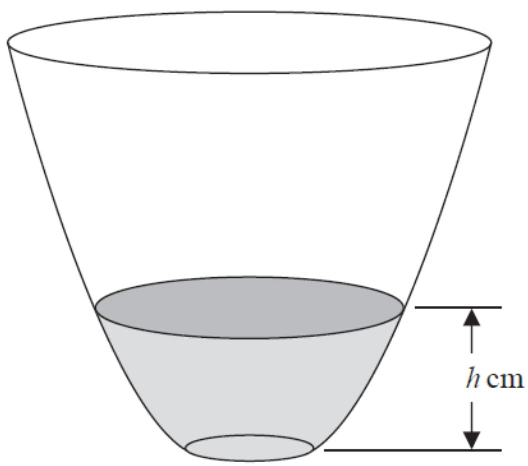


Figure 2

A vase with a circular cross-section is shown in Figure 2. Water is flowing into the vase.

When the depth of the water is h cm, the volume of water V cm^3 is given by

$$V = 4\pi h(h + 4), 0 \leq h \leq 25$$

Water flows into the vase at a constant rate of 80π cm^3 s^{-1}.

Find the rate of change of the depth of the water, in cm s^{-1}, when $h = 6$. **(5)**

(C4, June 2014, Q4)

23. (i) Find

$$\int xe^{4x}\,dx \qquad \textbf{(3)}$$

(ii) Find

$$\int \frac{8}{(2x-1)^3}\,dx, \qquad x > \frac{1}{2} \qquad \textbf{(2)}$$

(iii) Given that $y = \frac{\pi}{6}$ at $x = 0$, solve the differential equation

$$\frac{dy}{dx} = e^x \operatorname{cosec} 2y \operatorname{cosec} y \qquad \textbf{(7)}$$

(C4, June 2014, Q6)

24. At time t seconds the radius of a sphere is r cm, its volume is V cm^3 and its surface area is S cm^2.

[*You are given that* $V = \frac{4}{3}\pi r^3$ *and that* $S = 4\pi r^2$]

The volume of the sphere is increasing uniformly at a constant rate of 3 cm^3 s^{-1}.

(*a*) Find $\frac{dr}{dt}$ when the radius of the sphere is 4 cm, giving your answer to 3 significant figures.

(4)

(*b*) Find the rate at which the surface area of the sphere is increasing when the radius is 4 cm.

(2)

(C4, June 2014_R, Q5)

25. The rate of increase of the number, N, of fish in a lake is modelled by the differential equation

$$\frac{dN}{dt} = \frac{(kt-1)(5000-N)}{t}, \qquad t > 0,\ 0 < N < 5000$$

In the given equation, the time t is measured in years from the start of January 2000 and k is a positive constant.

(*a*) By solving the differential equation, show that

$$N = 5000 - Ate^{-kt}$$

where A is a positive constant.

(5)

After one year, at the start of January 2001, there are 1200 fish in the lake.

After two years, at the start of January 2002, there are 1800 fish in the lake.

(*b*) Find the exact value of the constant A and the exact value of the constant k.

(4)

(*c*) Hence find the number of fish in the lake after five years. Give your answer to the nearest hundred fish.

(1)

(C4, June 2014_R, Q7)

26. (*a*) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2}P(P-2)\cos 2t, \qquad t \geq 0,$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(*b*) solve this differential equation to show that

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$$

(7)

(*c*) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures.

(3)

(C4, June 2014, Q7)

27. (*a*) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.
The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2}P(P-2)\cos 2t, \qquad t \geq 0,$$

where P is the population in thousands, and t is the time measured in years since the start of the study.
Given that $P = 3$ when $t = 0$,
(*b*) solve this differential equation to show that

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$$

(7)

(*c*) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures.

(3)

(C4, June 2015, Q7)

28. The rate of decay of the mass of a particular substance is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{5}{2}x, \quad t \geq 0,$$

where x is the mass of the substance measured in grams and t is the time measured in days.

Given that $x = 60$ when $t = 0$,

(*a*) solve the differential equation, giving x in terms of t. You should show all steps in your working and give your answer in its simplest form.

(4)

(*b*) Find the time taken for the mass of the substance to decay from 60 grams to 20 grams. Give your answer to the nearest minute.

(3)

(C4, June 2016, Q4)

29.

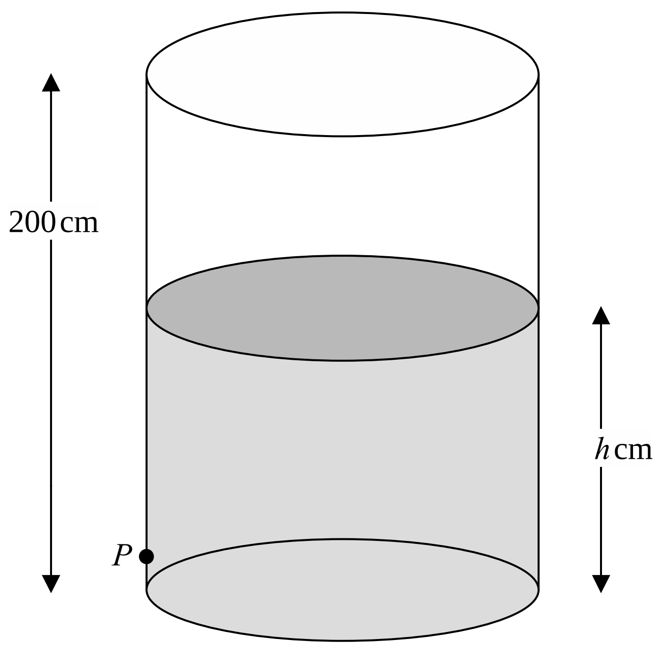


Diagram not drawn to scale

Figure 3

Figure 3 shows a vertical cylindrical tank of height 200 cm containing water.
Water is leaking from a hole P on the side of the tank.

At time t minutes after the leaking starts, the height of water in the tank is h cm.

The height h cm of the water in the tank satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(h-9)^{\frac{1}{2}}, \qquad 9 < h \leqslant 200$$

where k is a constant.

Given that, when $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute,

(a) find the value of k.

(2)

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$

(6)

(C4, June 2017, Q7)

30. (a) Use the substitution $u = 4 - \sqrt{x}$ to find

$$\int \frac{\mathrm{d}x}{4 - \sqrt{x}}$$

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{4 - \sqrt{h}}{20}$$

where h is the height in metres and t is the time measured in years after the tree is planted.

(b) Find the range in values for h for which the height of a tree in this species is increasing.

(2)

(c) Given that one of these trees is 1 metre high when it is planted, calculate the time it would take to reach a height of 10 metres. Write your answer to 3 significant figures.

(7)

(C34, IAL Jan 2014, Q9)

31. The volume V of a spherical balloon is increasing at a constant rate of 250 $\text{cm}^3\ \text{s}^{-1}$.
Find the rate of increase of the radius of the balloon, in cm s^{-1}, at the instant when the volume of the balloon is 12 000 cm^3.
Give your answer to 2 significant figures.

(5)

[*You may assume that the Volume V of a sphere of radius r is given by the formula* $V = \frac{4}{3}\pi r^3$.]

(C34, IAL Jun 2014, Q8)

32. (i) Given $x = \tan^2 4y$, $0 < y < \dfrac{\pi}{8}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a function of x.

Write your answer in the form $\dfrac{1}{A\left(x^p + x^q\right)}$, where A, p and q are constants to be found.

(5)

(ii) The volume V of a cube is increasing at a constant rate of 2 cm^3 s^{-1}. Find the rate at which the length of the edge of the cube is increasing when the volume of the cube is 64 cm^3.

(5)

(C34, IAL Jan 2015, Q6)

33. (*a*) Prove by differentiation that

$$\frac{\mathrm{d}}{\mathrm{d}y}(\ln \tan 2y) = \frac{4}{\sin 4y}, \qquad 0 < y < \frac{\pi}{4}$$

(4)

(*b*) Given that $y = \dfrac{\pi}{6}$ when $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x \sin 4y, \qquad 0 < y < \frac{\pi}{4}$$

Give your answer in the form $\tan 2y = A\mathrm{e}^{B\sin x}$, where A and B are constants to be determined.

(6)

(C34, IAL Jun 2015, Q8)

34.

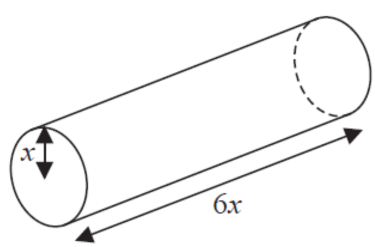


Figure 4

Figure 4 shows a right circular cylindrical rod which is expanding as it is heated.

At time t seconds the radius of the rod is x cm and the length of the rod is $6x$ cm.

Given that the **cross-sectional area** of the rod is increasing at a constant rate of $\dfrac{\pi}{20}$ cm^2 s^{-1}, find the rate of increase of the volume of the rod when $x = 2$.

Write your answer in the form $k\pi$ cm^3 s^{-1} where k is a rational number.

(6)

(C34, IAL Jun 2015, Q8)

35. (*a*) Express $\dfrac{3x^2-4}{x^2(3x-2)}$ in partial fractions.

(4)

(*b*) Given that $x > \dfrac{2}{3}$, find the general solution of the differential equation

$$x^2(3x-2)\frac{\mathrm{d}y}{\mathrm{d}x} = y(3x^2-4)$$

Give your answer in the form $y = \mathrm{f}(x)$.

(6)

(C34, IAL Jan 2016, Q9)

36.

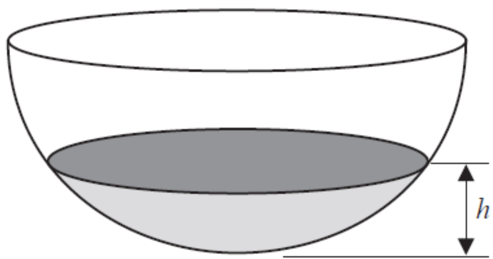


Figure 4

Figure 4 shows a hemispherical bowl containing some water.

At t seconds, the height of the water is h cm and the volume of the water is V cm^3, where

$$V = \frac{1}{3}\pi h^2(30-h), \qquad 0 < h \leq 10$$

The water is leaking from a hole in the bottom of the bowl.

Given that $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -\dfrac{1}{10}V$

(*a*) show that $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -\dfrac{h(30-h)}{30(20-h)}$

(5)

(*b*) Write $\dfrac{30(20-h)}{h(30-h)}$ in partial fraction form.

(3)

Given that $h = 10$ when $t = 0$,

(*c*) use your answers to parts (*a*) and (*b*) to find the time taken for the height of the water to fall to 5 cm. Give your answer in seconds to 2 decimal places.

(6)

(C34, IAL Jun 2016, Q13)

37. In freezing temperatures, ice forms on the surface of the water in a barrel. At time t hours after the start of freezing, the thickness of the ice formed is x mm. You may assume that the thickness of the ice is uniform across the surface of the water.

At 4pm there is no ice on the surface, and freezing begins.

At 6pm, after two hours of freezing, the ice is 1.5 mm thick.

In a simple model, the rate of increase of x, in mm per hour, is assumed to be constant for a period of 20 hours.

Using this simple model,

(a) express t in terms of x, **(2)**

(b) find the value of t when $x = 3$ **(1)**

In a second model, the rate of increase of x, in mm per hour, is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\lambda}{(2x+1)} \quad \text{where } / \text{ is a constant and } 0 \leq t \leq 20$$

Using this second model,

(c) solve the differential equation and express t in terms of x and / , **(3)**

(d) find the exact value for / , **(1)**

(e) find at what time the ice is predicted to be 3 mm thick. **(2)**

(C34, IAL Jan 2017, Q12)

38.

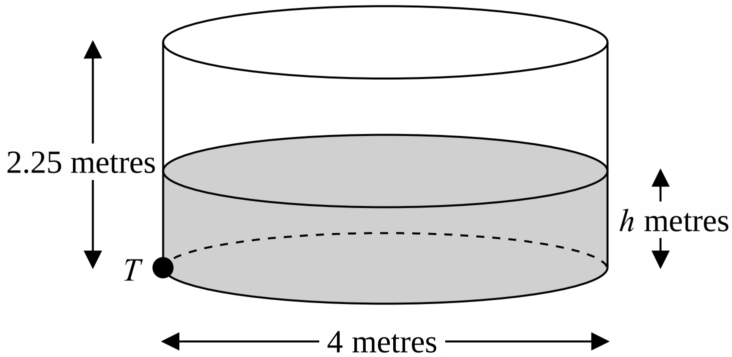


Figure 4

Figure 4 shows a right cylindrical water tank. The diameter of the circular cross section of the tank is 4 m and the height is 2.25 m. Water is flowing into the tank at a constant rate of 0.4π m^3 min^{-1}. There is a tap at a point T at the base of the tank. When the tap is open, water leaves the tank at a rate of $0.2\pi\sqrt{h}$ m^3 min^{-1}, where h is the height of the water in metres.

(*a*) Show that at time t minutes after the tap has been opened, the height h m of the water in the tank satisfies the differential equation

$$20\frac{\mathrm{d}h}{\mathrm{d}t} = 2 - \sqrt{h}$$

(5)

At the instant when the tap is opened, $t = 0$ and $h = 0.16$

(*b*) Use the differential equation to show that the time taken to fill the tank to a height of 2.25 m is given by

$$\int_{0.16}^{2.25} \frac{20}{2-\sqrt{h}}\,\mathrm{d}h$$

(2)

Using the substitution $h = (2 - x)^2$, or otherwise,

(*c*) find the time taken to fill the tank to a height of 2.25 m.

Give your answer in minutes to the nearest minute.

(*Solutions based entirely on graphical or numerical methods are not acceptable*.)

(7)

(C34, IAL Jun 2017, Q12)

39.

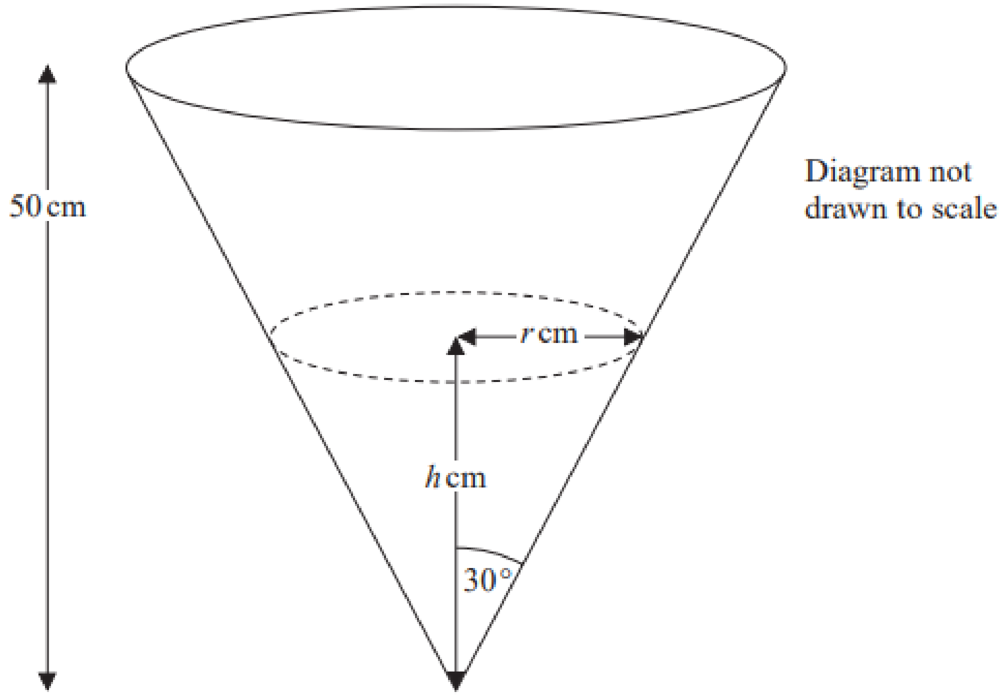


Figure 1

A water container is made in the shape of a hollow inverted right circular cone with semi-vertical angle of 30°, as shown in Figure 1. The height of the container is 50 cm.

When the depth of the water in the container is h cm, the surface of the water has radius r cm and the volume of water is V cm^3.

(a) Show that $V = \dfrac{1}{9}\pi h^3$

[*You may assume the formula* $V = \dfrac{1}{3}\pi r^2 h$ *for the volume of a cone.*]

(2)

Given that the volume of water in the container increases at a constant rate of 200 cm^3 s^{-1},

(b) find the rate of change of the depth of the water, in cm s^{-1}, when $h = 15$
Give your answer in its simplest form in terms of π.

(4)

(C4, Jun 2018, Q4)

40.

Given that $y = 2$ when $x = -\frac{\pi}{8}$, solve the differential equation

$$\frac{dy}{dx} = \frac{y^2}{3\cos^2 2x} \qquad -\frac{1}{2} < x < \frac{1}{2}$$

giving your answer in the form $y = f(x)$.

(6)

(C4, Jun 2018, Q6)

41. (*a*) Express $\frac{1}{(4-x)(2-x)}$ in partial fractions.

(2)

The mass, x grams, of a substance at time t seconds after a chemical reaction starts is modelled by the differential equation

$$\frac{dx}{dt} = k(4-x)(2-x), \qquad t \geqslant 0, \qquad 0 \leqslant x < 2$$

where k is a constant.

Given that when $t = 0$, $x = 0$

(*b*) solve the differential equation and show that the solution can be written as

$$x = \frac{4-4e^{2kt}}{1-2e^{2kt}}$$

(7)

Given that $k = 0.1$

(*c*) find the value of t when $x = 1$, giving your answer, in seconds, to 3 significant figures.

(2)

(C34, IAL June 2018, Q13)

42.

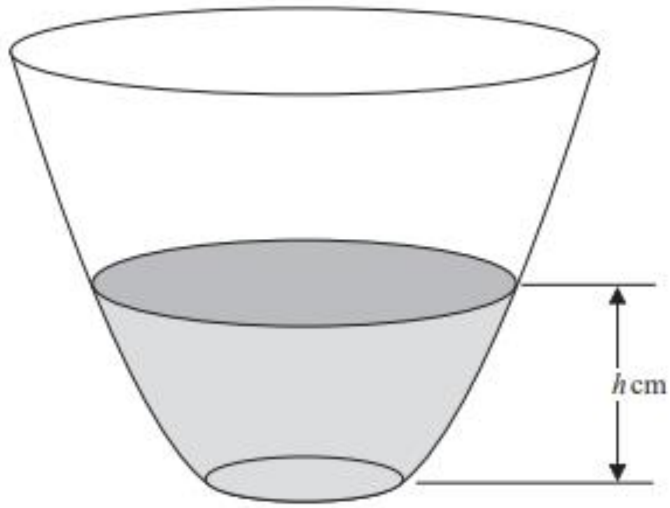


Figure 3

A container with a circular cross-section is shown in Figure 3.

Initially the container is empty. At time t seconds after water begins to flow into the container, the height of water in the container is h cm.

The height of water in the container satisfies the differential equation

$$\frac{dh}{dt} = \frac{k}{h+4} \qquad 0 \leqslant h \leqslant 35$$

where k is a constant.

When $h = 16$, the height of water in the container is increasing at a rate of $0.6\,\text{cm}\,\text{s}^{-1}$

(a) Find the value of k.

(1)

(b) Find the time taken to fill the container with water from empty to a height of 30 cm.

(4)

Given that the water flows into the container at a constant rate of $96\pi\,\text{cm}^3\,\text{s}^{-1}$

(c) find the volume of water in the container when $h = 30$
Give your answer in cm^3 to 3 significant figures.

(2)

(C4, Jun 2019, Q5)

43. (*a*) Given $0 \leqslant h < 25$, use the substitution $u = 5 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{5-\sqrt{h}} = -10\ln\left(5-\sqrt{h}\right) - 2\sqrt{h} + k$$

where k is a constant.

(6)

A team of scientists is studying a species of tree.

The rate of change in height of a tree of this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.2}\left(5-\sqrt{h}\right)}{5}$$

where h is the height of the tree in metres and t is the time in years after the tree is planted.

One of these trees is 2 metres high when it is planted.

(*b*) Use integration to calculate the time it would take for this tree to reach a height of 15 metres, giving your answer to one decimal place.

(7)

(*c*) Hence calculate the rate of change in height of this tree when its height is 15 metres. Write your answer in centimetres per year to the nearest centimetre.

(1)

(C34, IAL Oct 2017, Q11)

44. The volume of a spherical balloon of radius r cm is V cm^3, where $V = \frac{4}{3}\pi r^3$

(*a*) Find $\frac{dV}{dr}$

(1)

The volume of the balloon increases with time t seconds according to the formula

$$\frac{dV}{dt} = \frac{9000p}{(t+81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

(*b*) Using the chain rule, or otherwise, show that

$$\frac{dr}{dt} = \frac{k}{r^n(t+81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

where k and n are constants to be found.

(2)

Initially, the radius of the balloon is 3 cm.

(*c*) Using the values of k and n found in part (*b*), solve the differential equation

$$\frac{dr}{dt} = \frac{k}{r^n(t+81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

to obtain a formula for r in terms of t.

(6)

(*d*) Hence find the radius of the balloon when $t = 175$, giving your answer to 3 significant figures.

(1)

(*e*) Find the rate of increase of the radius of the balloon when $t = 175$. Give your answer to 3 significant figures.

(2)

(C34, IAL Jan 2018, Q10)

45. The volume of a spherical balloon of radius r m is Vm^3, where $V = \frac{4}{3}\pi r^3$

(a) Find $\frac{\mathrm{d}V}{\mathrm{d}r}$

(1)

Given that the volume of the balloon increases with time t seconds according to the formula

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{20}{V(0.05t+1)^3}, \qquad t \geqslant 0$$

(b) find an expression in terms of r and t for $\frac{\mathrm{d}r}{\mathrm{d}t}$

(3)

Given that $V = 1$ when $t = 0$

(c) solve the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{20}{V(0.05t+1)^3}$$

giving your answer in the form $V^2 = \mathrm{f}(t)$.

(6)

(d) Hence find the radius of the balloon at time $t = 20$, giving your answer to 3 significant figures.

(3)

(C34, IAL Oct 2018, Q13)

46. The height, h metres, of a shrub, t years after it was planted, is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{2h^{\frac{3}{2}}}{5t^2} \qquad t > 0$$

(a) Given that $h = 1$ when $t = 1$, show that

$$h = \frac{at^2}{(1+bt)^2}$$

where a and b are constants to be found.

(7)

(b) Hence find, according to the model, the limit of the height of the shrub.

(2)

(C34, IAL Nov 2019, Q5)

47.

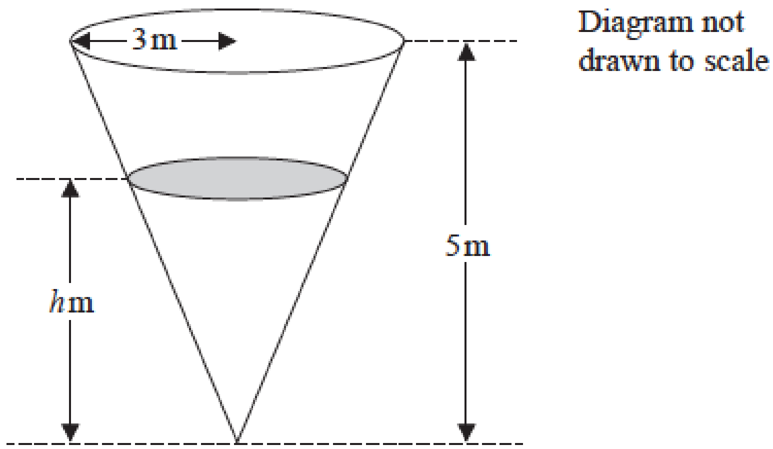


Figure 3

Figure 3 shows a container in the shape of an inverted right circular cone which contains some water.

The cone has an internal radius of 3 m and a vertical height of 5 m as shown in Figure 3.

At time t seconds, the height of the water is h metres, the volume of the water is V m^3 and water is leaking from a hole in the bottom of the container at a constant rate of 0.02 m^3 s^{-1}

[*The volume of a cone of radius r and height h is* $\frac{1}{3}\pi r^2 h$.]

(*a*) Show that, while the water is leaking,

$$h^2 \frac{dh}{dt} = -\frac{1}{k\pi}$$

where k is a constant to be found.

(5)

Given that the container is initially full of water,

(*b*) express h in terms of t.

(3)

(*c*) Find the time taken for the container to empty, giving your answer to the nearest minute.

(2)

(C34, IAL Jan 2019, Q10)