OCR Core Maths 3

Past paper questions Differentiations

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Differentiation

- The basic building blocks for differentiation (that we know at present) are:

$$y = ax^n \qquad y = e^x \qquad y = \ln x$$
$$\frac{dy}{dx} = anx^{n-1}, \qquad \frac{dy}{dx} = e^x, \qquad \frac{dy}{dx} = \frac{1}{x}.$$

  Also we know the idea that (for $f(x) \equiv f$, $g(x) \equiv g$ and $k =$ constant)

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g) \quad \text{and} \quad \frac{d}{dx}(kf) = k\frac{d}{dx}(f).$$

  (In big-boy speak we say that $\frac{d}{dx}$ is a linear operator.)

- The *chain rule* is incredibly important! It states that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

  This seems obvious from the way that differentials are written, but remember that they should not be thought of as fractions.

- If a bit of a $y = \ldots$ is making the differentiation difficult, then ask yourself the question "would making the complicated bit u make it easier for me to deal with?" For example with $y = (2x-5)^{20}$ the function would be considerably easier if $u = 2x - 5$ because y becomes $y = u^{20}$. Similarly with $y = e^{x^2+1}$ my life would be easier if $u = x^2 + 1$ because y would become $y = e^u$.

- It can be applied as follows to the example $y = (x^4 + x)^{10}$. Let $u = x^4 + x$, so

$$y = u^{10} \qquad u = x^4 + x$$
$$\frac{dy}{du} = 10u^9 \qquad \frac{du}{dx} = 4x^3 + 1.$$

  Therefore $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 10u^9 \times (4x^3+1) = 10(4x^3+1)(x^4+x)^9$.

- The above method works all the time but it is a little slow. You will notice the general result that if $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$. So we can just write down the answer to similar problems. For example if $y = (3x^2+1)^5$ then $\frac{dy}{dx} = 30x(3x^2+1)^4$.

- We can also combine the chain rule with exponentials and logarithms to gain the following important results:

$$\begin{aligned}
\frac{d}{dx}(e^{ax}) &= ae^{ax} && \text{using } u = ax \\
\frac{d}{dx}(e^{f(x)}) &= f'(x)e^{f(x)} && \text{using } u = f(x) \\
\frac{d}{dx}(\ln ax) &= \frac{a}{ax} = \frac{1}{x} && \text{using } u = ax \\
\frac{d}{dx}(\ln f(x)) &= \frac{f'(x)}{f(x)} && \text{using } u = f(x).
\end{aligned}$$

- The *product rule* states that when $y = u \times v$ (where u and v are functions of x) we can differentiate it using the product rule. It states that

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

For example if $y = x^2(x^3-1)^3$ then

$$\begin{aligned}
\frac{dy}{dx} &= [2x \times (x^3-1)^3] + [x^2 \times 3(x^3-1)^2 \times 3x^2] \\
&= 2x(x^3-1)^3 + 9x^4(x^3-1)^2 \\
&= x(x^3-1)^2[2(x^3-1) + 9x^3] \\
&= x(x^3-1)^2(11x^3-2).
\end{aligned}$$

- With the product rule you often end up with expressions such as

$$\frac{dy}{dx} = 2x^3(2x+1)^{-4} - x^4(2x+1)^{-5}.$$

When tidying these things up you must pull out (as always) *the lowest power* of any common elements even if they are negative or fractional; here we have x^3 and $(2x+1)^{-5}$:

$$\begin{aligned}
\frac{dy}{dx} &= 2x^3(2x+1)^{-4} - x^4(2x+1)^{-5} \\
&= x^3(2x+1)^{-5}[2(2x+1) - x] \\
&= \frac{x^3(3x+2)}{(2x+1)^5}.
\end{aligned}$$

- Very similar to the product rule is the *quotient rule*. It is used for functions of the form $y = \frac{u}{v}$. It states

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

For example differentiating $y = \frac{x^3}{x^2+1}$ gives

$$\frac{dy}{dx} = \frac{(x^2+1) \times 3x^2 - x^3 \times 2x}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2}.$$

- Once again, although $\frac{dy}{dx}$ is not a fraction, it can be treated as such when taking its reciprocal, so
$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}.$$
For example if you have $V = \frac{4}{3}\pi r^3$ then $\frac{dV}{dr} = 4\pi r^2$ and also $\frac{dr}{dV} = \frac{1}{4\pi r^2}$. This idea most useful in the topic of...

- ...*connected rates of change.* Here you need to use the chain rule to 'connect' differentials you know to get one you need. Questions mostly ask you for $\frac{dy}{dx}$ (say) and you need to find a third variable to construct $\frac{dy}{dx} = \frac{dy}{d...} \times \frac{d...}{dx}$ by the chain rule. For example: The area A of a circle is increasing a rate of 3cm^2/s, find the rate at which the radius r is increasing when $r = 20$cm. We want to find $\frac{dr}{dt}$ so
$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{dA} \times \frac{dA}{dt} \qquad \text{but} \qquad A = \pi r^2, \text{ so } \frac{dA}{dr} = 2\pi r. \\ &= \frac{1}{2\pi r} \times 3 \\ &= \frac{3}{40\pi}. \end{aligned}$$

1.

(a) Find the exact value of the x-coordinate of the stationary point of the curve $y = x \ln x$. [4]

(b) The equation of a curve is $y = \frac{4x + c}{4x - c}$, where c is a non-zero constant. Show by differentiation that this curve has no stationary points. [3]

Q6 June 2005

2.

(a) Differentiate $x^2(x + 1)^6$ with respect to x. [3]

(b) Find the gradient of the curve $y = \frac{x^2 + 3}{x^2 - 3}$ at the point where $x = 1$. [3]

Q3 Jan 2006

3.

Find the equation of the tangent to the curve $y = \sqrt{4x + 1}$ at the point $(2, 3)$. [5]

Q1 June 2006

4.

It is given that $y = 5^{x-1}$.

(i) Show that $x = 1 + \frac{\ln y}{\ln 5}$. [2]

(ii) Find an expression for $\frac{dx}{dy}$ in terms of y. [2]

(iii) Hence find the exact value of the gradient of the curve $y = 5^{x-1}$ at the point $(3, 25)$. [2]

Q4 June 2006

5.

Find the equation of the tangent to the curve $y = \frac{2x + 1}{3x - 1}$ at the point $\left(1, \frac{3}{2}\right)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. [5]

Q1 Jan 2007

6.

(i) Given that $x = (4t + 9)^{\frac{1}{2}}$ and $y = 6e^{\frac{1}{2}x+1}$, find expressions for $\frac{dx}{dt}$ and $\frac{dy}{dx}$. [4]

(ii) Hence find the value of $\frac{dy}{dt}$ when $t = 4$, giving your answer correct to 3 significant figures. [3]

Q4 Jan 2007

7.

Differentiate each of the following with respect to x.

(i) $x^3(x+1)^5$ [2]

(ii) $\sqrt{3x^4 + 1}$ [3]

Q1 June 2007

8.

Earth is being added to a pile so that, when the height of the pile is h metres, its volume is V cubic metres, where

$$V = (h^6 + 16)^{\frac{1}{2}} - 4.$$

(i) Find the value of $\frac{dV}{dh}$ when $h = 2$. [3]

(ii) The volume of the pile is increasing at a constant rate of 8 cubic metres per hour. Find the rate, in metres per hour, at which the height of the pile is increasing at the instant when $h = 2$. Give your answer correct to 2 significant figures. [3]

Q4 Jan 2008

9.

A curve has equation $y = \frac{xe^{2x}}{x+k}$, where k is a non-zero constant.

(i) Differentiate xe^{2x}, and show that $\frac{dy}{dx} = \frac{e^{2x}(2x^2 + 2kx + k)}{(x+k)^2}$. [5]

(ii) Given that the curve has exactly one stationary point, find the value of k, and determine the exact coordinates of the stationary point. [5]

Q7 Jan 2008

10.

Find, in the form $y = mx + c$, the equation of the tangent to the curve

$$y = x^2 \ln x$$

at the point with x-coordinate e. [6]

Q3 June 2008

11.

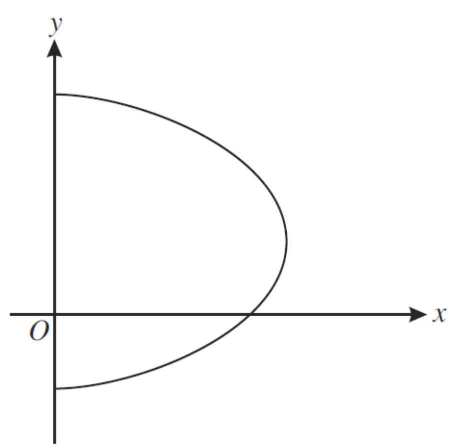
For each of the following curves, find $\frac{dy}{dx}$ and determine the exact x-coordinate of the stationary point:

(i) $y = (4x^2 + 1)^5$, **[3]**

(ii) $y = \frac{x^2}{\ln x}$. **[4]**

Q4 Jan 2009

12.



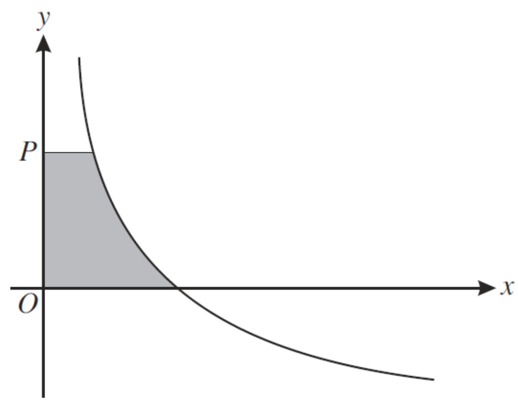
The diagram shows the curve with equation $x = (37 + 10y - 2y^2)^{\frac{1}{2}}$.

(i) Find an expression for $\frac{dx}{dy}$ in terms of y. **[2]**

(ii) Hence find the equation of the tangent to the curve at the point (7, 3), giving your answer in the form $y = mx + c$. **[5]**

Q6 June 2009

13.



The diagram shows the curve with equation

$$y = \frac{6}{\sqrt{x}} - 3.$$

The point P has coordinates $(0, p)$. The shaded region is bounded by the curve and the lines $x = 0$, $y = 0$ and $y = p$. The shaded region is rotated completely about the y-axis to form a solid of volume V.

(i) Show that $V = 16\pi\left(1 - \frac{27}{(p+3)^3}\right)$. **[6]**

(ii) It is given that P is moving along the y-axis in such a way that, at time t, the variables p and t are related by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \tfrac{1}{3}p + 1.$$

Find the value of $\frac{\mathrm{d}V}{\mathrm{d}t}$ at the instant when $p = 9$. **[4]**

Q8 Jan 2009

14.

(a) Show that, for all non-zero values of the constant k, the curve

$$y = \frac{kx^2 - 1}{kx^2 + 1}$$

has exactly one stationary point. **[5]**

(b) Show that, for all non-zero values of the constant m, the curve

$$y = \mathrm{e}^{mx}(x^2 + mx)$$

has exactly two stationary points. **[7]**

Q9 June 2009

15.

The equation of a curve is $y = (x^2 + 1)^8$.

(i) Find an expression for $\frac{dy}{dx}$ and hence show that the only stationary point on the curve is the point for which $x = 0$. **[4]**

(ii) Find an expression for $\frac{d^2y}{dx^2}$ and hence find the value of $\frac{d^2y}{dx^2}$ at the stationary point. **[5]**

Q5 Jan 2010

16.

Find $\frac{dy}{dx}$ in each of the following cases:

(i) $y = x^3e^{2x}$, **[2]**

(ii) $y = \ln(3 + 2x^2)$, **[2]**

(iii) $y = \frac{x}{2x+1}$. **[2]**

Q1 June 2010

17.

A giant spherical balloon is being inflated in a theme park. The radius of the balloon is increasing at a rate of 12 cm per hour. Find the rate at which the surface area of the balloon is increasing at the instant when the radius is 150 cm. Give your answer in cm^2 per hour correct to 2 significant figures.

[Surface area of sphere = $4\pi r^2$.] **[3]**

Q3 Jan 2011

18.

The equation of a curve is $y = x^2 \ln(4x - 3)$. Find the exact value of $\frac{d^2y}{dx^2}$ at the point on the curve for which $x = 2$. **[8]**

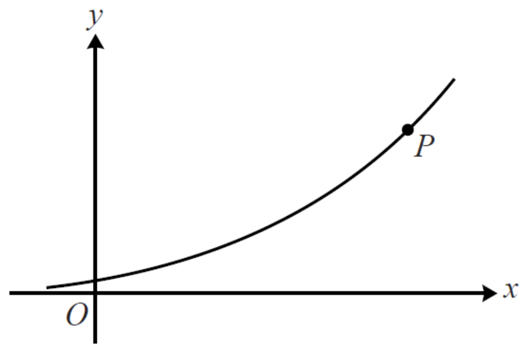
Q5 June 2011

19.

Find the equation of the normal to the curve $y = \frac{x^2+4}{x+2}$ at the point $(1, \frac{5}{3})$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. **[7]**

Q3 Jan 2012

20.



The diagram shows the curve with equation $x = \ln(y^3 + 2y)$. At the point P on the curve, the gradient is 4 and it is given that P is close to the point with coordinates (7.5, 12).

(i) Find $\frac{dx}{dy}$ in terms of y. **[2]**

(ii) Show that the y-coordinate of P satisfies the equation

$$y = \frac{12y^2 + 8}{y^2 + 2}.$$ **[3]**

(iii) By first using an iterative process based on the equation in part **(ii)**, find the coordinates of P, giving each coordinate correct to 3 decimal places. **[5]**

Q6 Jan 2012

21.

The volume, V m^3, of liquid in a container is given by

$$V = (3h^2 + 4)^{\frac{3}{2}} - 8,$$

where h m is the depth of the liquid.

(i) Find the value of $\frac{dV}{dh}$ when $h = 0.6$, giving your answer correct to 2 decimal places. **[4]**

(ii) Liquid is leaking from the container. It is observed that, when the depth of the liquid is 0.6 m, the depth is decreasing at a rate of 0.015 m per hour. Find the rate at which the volume of liquid in the container is decreasing at the instant when the depth is 0.6 m. **[3]**

Q6 June 2012

22.

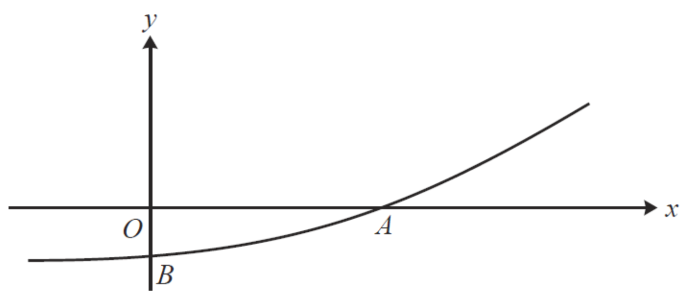
For each of the following curves, find the gradient at the point with x-coordinate 2.

(i) $y = \dfrac{3x}{2x + 1}$ **[3]**

(ii) $y = \sqrt{4x^2 + 9}$ **[3]**

Q1 Jan 2013

23.



The diagram shows the curve with equation

$$x = (y + 4)\ln(2y + 3).$$

The curve crosses the x-axis at A and the y-axis at B.

(i) Find an expression for $\dfrac{dx}{dy}$ in terms of y. **[3]**

(ii) Find the gradient of the curve at each of the points A and B, giving each answer correct to 2 decimal places. **[5]**

Q7 Jan 2013

24.

Find the exact value of the gradient of the curve

$$y = \sqrt{4x - 7} + \frac{4x}{2x + 1}$$

at the point for which $x = 4$. **[6]**

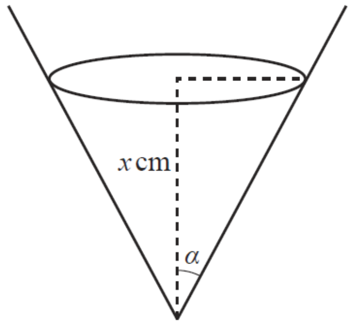
Q4 June 2013

25.

Given that $y = 4x^2 \ln x$, find the value of $\dfrac{d^2y}{dx^2}$ when $x = e^2$. **[5]**

Q1 June 2014

26.



The diagram shows a container in the form of a right circular cone. The angle between the axis and the slant height is α, where $\alpha = \tan^{-1}(\frac{1}{2})$. Initially the container is empty, and then liquid is added at the rate of $14\,\text{cm}^3$ per minute. The depth of liquid in the container at time t minutes is x cm.

(i) Show that the volume, $V\,\text{cm}^3$, of liquid in the container when the depth is x cm is given by

$$V = \frac{1}{12}\pi x^3.$$

[The volume of a cone is $\frac{1}{3}\pi r^2 h$.] **[2]**

(ii) Find the rate at which the depth of the liquid is increasing at the instant when the depth is 8 cm. Give your answer in cm per minute correct to 2 decimal places. **[3]**

Q3 June 2013

26.

Find the equation of the tangent to the curve $y = \frac{5x+4}{3x-8}$ at the point $(2, -7)$. **[5]**

Q1 June 2015

27.

The volume, V cubic metres, of water in a reservoir is given by

$$V = 3(2+\sqrt{h})^6 - 192,$$

where h metres is the depth of the water. Water is flowing into the reservoir at a constant rate of 150 cubic metres per hour. Find the rate at which the depth of water is increasing at the instant when the depth is 1.4 metres. **[5]**

Q3 June 2015