Kumar's Maths Revision

Further Pure 1
Complex numbers

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Complex Numbers

The EDEXCEL syllabus says that candidates should:

a) understand the idea of a complex number, recall the meaning of the terms *real part*, *imaginary part*, *modulus*, *argument*, *conjugate*, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal;

b) be able to carry out operations of *addition*, *subtraction*, *multiplication* and *division* of two complex numbers;

c) be able to use the result that, for a polynomial equation with real coefficients, any non-real roots occur in *conjugate pairs*;

d) be able to represent complex numbers geometrically by means of an *Argand diagram*, and understand the geometrical effects of conjugating a complex number and of adding and subtracting two complex numbers;

e) find the two square roots of a complex number;

Section 1: Introduction to complex numbers

Suppose we wished to solve the quadratic equation $x^2 + 4x + 5 = 0$.
Using the quadratic formula, the solutions would be:

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2}.$$

We notice a problem however since $\sqrt{-4}$ is not a real number. So the equation $x^2 + 4x + 5 = 0$ does not have any real roots.

However, suppose we introduced the symbol i to represent $\sqrt{-1}$. We could then find expressions for the solutions of the quadratic:

$$x = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm \sqrt{4 \times -1}}{2} = \frac{-4 \pm 2i}{2}.$$

So the equation has two solutions: $x = -2 + i$ or $x = -2 - i$.

These two solutions are called **complex numbers**.

1.1 Some definitions

Suppose that z is a complex number. Let $z = a + ib$.
The **real part** of z, written Re(z), is a.
The **imaginary part** of z, written Im(z), is b.
The **complex conjugate** of z, written z^* or $\bar{z}$, is $z^* = a - ib$.

Example 1: Let $z = 5 - 2i$, $w = -2 + i$ and $u = 7i$.
Then:

Re(z) = 5	Im(z) = -2	$z^* = 5 + 2i$
Re(w) = -2	Im(w) = 1	$w^* = -2 - i$.
Re(u) = 0	Im(u) = 7	$u^* = -7i$.

1.2 Solving quadratic equations

Example 2: Solve the quadratic equation $2x^2 + 4x + 5 = 0$, giving your answers as complex numbers in surd form.

Solution: Using the quadratic formula:

$$x = \frac{-4 \pm \sqrt{16 - 4 \times 2 \times 5}}{4} = \frac{-4 \pm \sqrt{-24}}{4}$$

$$= \frac{-4 \pm i\sqrt{24}}{4} = \frac{-4 \pm 2\sqrt{6}\,i}{4}$$

So the solutions are $x = -1 + \frac{\sqrt{6}}{2}i$ or $x = -1 - \frac{\sqrt{6}}{2}i$.

Notice that the two solutions are **complex conjugates** of each other. The solutions form a **conjugate pair**. This leads to this very important result:

Consider the equation $ax^2 + bx + c = 0$ where a, b and c are **real numbers**. If the equation has complex roots, then the two roots are always **conjugates** of each other.

Note: If a quadratic equation has any complex coefficients then this result doesn't apply.

Example: If $z = 1 - i$ is one solution of the quadratic equation $z^2 - 2z + 2 = 0$, then the second solution must be the **complex conjugate** (as the quadratic has real coefficients). So the second solution is $z = 1 + i$.

Section 2: Calculating with complex numbers

2.1 Adding and subtracting

Two complex numbers are added or subtracted by collecting together their real and imaginary parts.

So

$$(x+iy)+(u+iv)=(x+u)+i(y+v)$$

and

$$(x+iy)-(u+iv)=(x-u)+i(y-v)$$

We can also easily multiply a complex number by a real number:

$$k(x+iy)=kx+iky$$

Example: If $z = 4 + 2i$ and $w = 3 - i$, then

$z + w = (4 + 2i) + (3 - i) = 7 + i$

$z - w = (4 + 2i) - (3 - i) = 1 + 3i$ (being careful with the negative signs!)

$3z + 2w = 3(4 + 2i) + 2(3 - i) = (12 + 6i) + (6 - 2i) = 18 + 4i$

$2w - z^* = 2(3 - i) - (4 - 2i) = (6 - 2i) - (4 - 2i) = 2$

2.2 Multiplying

Complex numbers can be multiplied using the general method for expanding brackets.

Examples:

$(2 + 5i)(4 - 3i) = 8 - 6i + 20i - 15i^2$ Remember: $i^2 = -1$
$= 8 - 6i + 20i - 15(-1)$
$= 23 + 14i$

$(3 + 2i)^2 = (3 + 2i)(3 + 2i) = 9 + 6i + 6i + 4i^2$
$= 9 + 12i + 4(-1)$
$= 5 + 12i$

2.3 Dividing

To divide complex conjugates, you multiply through by the complex conjugate of the denominator:

Example: If $z = 3 - i$ and $w = 1 - 2i$, then

$$z \div w = \frac{3-i}{1-2i}$$

$$= \frac{3-i}{1-2i} \times \frac{1+2i}{1+2i} \qquad \text{(as the complex conjugate of } w \text{ is } w^* = 1 + 2i\text{)}$$

$$= \frac{3+6i-i-2i^2}{1+2i-2i-4i^2} = \frac{3+5i-2(-1)}{1-4(-1)} \qquad \text{(as } i^2 = -1\text{)}$$

$$= \frac{5+5i}{5}$$

Therefore:

$$z \div w = 1 + i$$

Note: When a complex number is multiplied by its complex conjugate the answer is always purely a real number.
To show this, suppose $z = x + iy$.
Then $zz^* = (x + iy)(x - iy) = x^2 - 2ixy + 2ixy - i^2y^2$
$= x^2 + y^2$

Worked examination question:

a) Express in the form $a + ib$,
   (i) $(3 + i)^2$
   (ii) $(2 + 4i)(3 + i)$.

b) The quadratic equation

$$z^2 - (2+4i)z + 8i - 6 = 0$$

has roots z_1 and z_2.
   i) Verify that $z_1 = 3 + i$ is a root of the equation.
   ii) By considering the coefficients of the quadratic, write down the sum of the roots;
   iii) Explain why z_1^*, the complex conjugate of z_1, is **not** a root of the quadratic;
   iv) Find the other root, z_2, in the form $a + ib$.

Solution:

a) (i) $(3 + i)^2 = (3 + i)(3 + i) = 9 + 3i + 3i + i^2$
$= 9 + 6i - 1 = 8 + 6i$ Remember: $i^2 = -1$

(ii) $(2 + 4i)(3 + i) = 6 + 2i + 12i + 4i^2$
$= 6 + 14i - 4 = 2 + 14i$

b) (i) We substitute $z = 3 + i$ into the expression $z^2 - (2+4i)z + 8i - 6$ to check that it gives an answer of 0.

$(3 + i)^2 - (2 + 4i)(3 + i) + 8i - 6$
$= 8 + 6i - (2 + 14i) + 8i - 6$ (using the results from (a))
$= 8 + 6i - 2 - 14i + 8i - 6$
$= 0 + 0i = 0$ (as required)

(ii) In a quadratic equation, the sum of roots is given by the expression $-b/a$.
So here the sum of the roots is $(2 + 4i)$ (since $b = -(2 + 4i)$ and $a = 1$)

(iii) In a quadratic equation with real coefficients, any complex roots form a conjugate pair. However, this quadratic does not have real coefficients so the roots are not complex conjugates of each other.

(iv) From (ii), $z_1 + z_2 = 2 + 4i$
So $3 + i + z_2 = 2 + 4i$
i.e. $z_2 = 2 + 4i - (3 + i) = -1 + 3i$.

Examination Question:

Given that $z = \dfrac{1+i}{1-2i}$, find z in the form $a + ib$.

Examination Question 2:

$z_1 = -3 + 4i$ $\qquad$ $z_2 = 1 + 2i$

Express $z_1 z_2$ and $\dfrac{z_1}{z_2}$ each in the form $a + ib$, where $a, b \in \mathbb{R}$.

Examination Question 3:

The complex numbers z and w are such that

$$z = -2 + 5i \qquad zw = 14 + 23i$$

Find w in the form $p + qi$, where p and q are real.

Examination Question 4:

Given that $z = -2 + 2\sqrt{3}i$, show that $z^2 + 4z$ is real.

Examination Question 5

(a) Show that $(3 - i)^2 = 8 - 6i$.

(b) The quadratic equation

$$az^2 + bz + 10i = 0,$$

where a and b are real, has a root $3 - i$.

(i) Show that a = 3 and find the value of b.

(ii) Determine the other root of the equation, giving your answer in the form $p + iq$.

(Hint for (ii): Use the fact that the sum of the roots in any quadratic is –b/a).

2.4 Equivalence of two complex numbers

Two complex numbers are equal to each other if and only if their real parts are equal and their imaginary parts are equal, i.e

If $z = x + iy$ and $w = u + iv$, then $z = w$ if and only if $x = u$ and $y = v$.

2.5 Solving linear equations with complex coefficients

Linear equations can be solved by substituting $z = x + iy$.

Example 1: Solve $4z - 2 + 5i = 6 - 7i$

Solution: Let $z = x + iy$.

Then: $4(x + iy) - 2 + 5i = 6 - 7i$

$4x + 4iy - 2 + 5i = 6 - 7i$ (expanding bracket)

$(4x - 2) + (4y + 5)i = 6 - 7i$

Therefore, comparing real and imaginary parts:

$4x - 2 = 6$ i.e. $x = 2$

$4y + 5 = -7$ i.e. $y = -3$

So the solution to the original equation must be $z = 2 - 3i$.

Example 2: Find z when $2z - 5z^* = 9 + 14i$.

Solution: Let $z = x + iy$. Then $z^* = x - iy$.

So:

$2(x + iy) - 5(x - iy) = 9 + 14i$

$2x + 2iy - 5x + 5iy = 9 + 14i$

i.e. $-3x + 7iy = 9 + 14i$.

Comparing real and imaginary parts, we see that $x = -3$ and $y = 2$.

So the solution is $z = -3 + 2i$.

Example 3: Solve $(4 + 2i)z + (3 - 2i) = 9 - 4i$

Solution: Write $z = x + iy$.

Then $(4+2i)(x+iy)+(3-2i)=9-4i$

So $(4x+4iy+2ix+2i^2y)+(3-2i)=9-4i$

i.e. $(4x-2y+4iy+2ix)+(3-2i)=9-4i$ (using $i^2 = -1$)

Collecting the real and imaginary terms together on the left hand side:

$$(4x-2y+3)+i(4y+2x-2)=9-4i$$

Comparing real and imaginary parts on both sides, we get the equations:

$4x - 2y + 3 = 9$ i.e. $4x - 2y = 6$ or $2x - y = 3$ (1)

and $4y + 2x - 2 = -4$ i.e. $2x + 4y = -2$ or $x + 2y = -1$ (2)

Equations (1) and (2) can be solved simultaneously:

$5x = 5$ $(2\times(1) + (2))$

i.e. $x = 1$

From equation (1), $y = 2x - 3 = 2 - 3 = -1$.

Therefore the solution to the equation is $z = 1 - i$.

Alternative solution:

The equation to be solved is $(4 + 2i)z + (3 - 2i) = 9 - 4i$

So … $(4 + 2i)z = (9 - 4i) - (3 - 2i)$

$= 6 - 2i$

Therefore

$$z = \frac{6-2i}{4+2i} = \frac{6-2i}{4+2i} \times \frac{4-2i}{4-2i}$$

So $$z = \frac{24-12i-8i-4}{16+4} = \frac{20-20i}{20}$$

This gives the solution $z = 1 - i$

Worked examination style question

It is given that $z = x + iy$ and that z^* is the complex conjugate of z.

a) Express $2z - 3z^*$ in the form $p + qi$.

b) Find the value of z for which $2z - 3z^* = -5 + 15i$

Solution:

If $z = x + iy$, then $z^* = x - iy$.

a) So $2z - 3z^* = 2(x + iy) - 3(x - iy) = 2x + 2iy - 3x + 3iy = -x + 5iy$.

b) If $2z - 3z^* = -5 + 15i$

then …. $-x + 5iy = -5 + 15i$

So … $x = 5$ and $y = 3$.

Examination style question:

Given that $(3 - 2i)(x + 5i) = y + 11i$

where x and y are real numbers:

a) find two equations for x and y;

b) find the values of x and y.

Examination Question 1

It is given that $z = x + iy$ and that z^* is the complex conjugate of z.

a) Express $z + 2z^*$ in the form $p + qi$.

b) Find the value of z for which $z + 2z^* = 9 + 2i$.

Examination Question 2:

It is given that $z = x + iy$, where x and y are real numbers.

a) Write down, in terms of x and y, an expression for z^*, the complex conjugate of z.

b) Find, in terms of x and y, the real and imaginary parts of

$$2z - iz^*$$

c) Find the complex number z such that

$$2z - iz^* = 3i.$$

Examination question 3:

(a) (i) Calculate $(2+i\sqrt{5})(\sqrt{5}-i)$.

(ii) Hence verify that $\sqrt{5}-i$ is a root of the equation

$$(2+i\sqrt{5})z = 3z^*$$

where z^* is the complex conjugate of z.

(b) The quadratic equation

$$x^2 + px + q = 0$$

in which the coefficients p and q are real, has a complex root $\sqrt{5}-i$.

(i) Write down the other root of the equation.

(ii) Find the sum and the product of the two roots of the equation.

Examination question 4:
Solve the simultaneous equations

$$iz + 2x = 1$$
$$z - (1 + i)w = i$$

giving your answers for z and w in the form $a + ib$.

2.6 Finding the square root of a complex number

One particular more difficult problem that occasionally occurs in FP1 examinations is to find the square root of a complex number. You should be prepared in case it comes up in your exam!!

Example: Find the square roots of 9 – 12i.

Solution: Let $a + bi$ be a square root of $9 - 12i$, where a and b are real numbers.
Then $(a + bi)^2 = 9 - 12i$.
So:

$$(a + bi)(a + bi) = 9 - 12i$$

i.e. $a^2 + abi + abi + b^2i^2 = 9 - 12\text{i}$

i.e. $a^2 - b^2 + 2abi = 9 - 12i$.

Comparing real and imaginary parts, we get two equations:

$a^2 - b^2 = 9$ ①

and $ab = -6$ ②

From equation ②, we get $b = \dfrac{-6}{a}$.

Substituting this into ① gives: $a^2 - \left(\dfrac{-6}{a}\right)^2 = 9$

i.e. $a^2 - \dfrac{36}{a^2} = 9$

i.e. $a^4 - 36 = 9a^2$

or $a^4 - 9a^2 - 36 = 0$

This is a quadratic equation in a^2. It can be factorised:

$$(a^2 - 12)(a^2 + 3) = 0$$

So $a^2 = 12$ or $a^2 = -3$.

As a is real, we must have $a = \pm\sqrt{12} = \pm 2\sqrt{3}$.

If $a = 2\sqrt{3}$, then $b = \dfrac{-6}{a} = \dfrac{-6}{2\sqrt{3}} = -\sqrt{3}$.

If $a = -2\sqrt{3}$, then $b = \dfrac{-6}{a} = \dfrac{-6}{-2\sqrt{3}} = \sqrt{3}$.

Therefore the square roots of $9 - 12i$ are $2\sqrt{3} - \sqrt{3}i$ and $-2\sqrt{3} + \sqrt{3}i$.

Note: All questions about finding the square root of a complex number can be solved using the same method.

Examination question:
Find the roots of the equation $z^2 = 21 - 20i$.

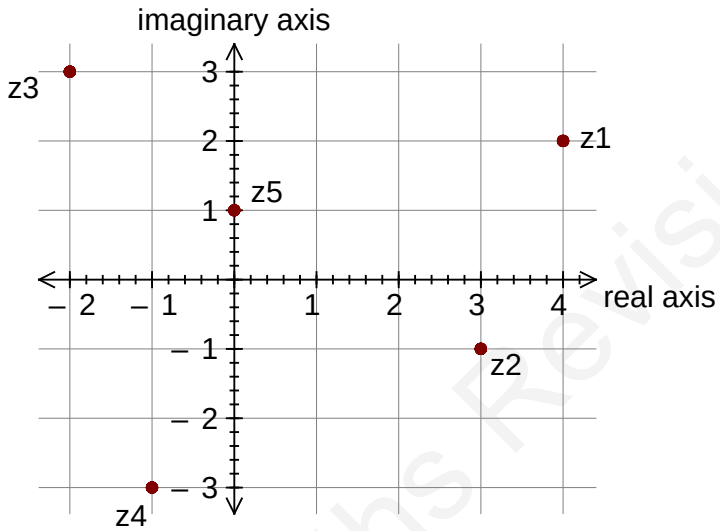
Hint: Remember to put $z = a + bi$.

Section 3: Argand Diagrams

3.1 Representing complex numbers on Argand diagrams

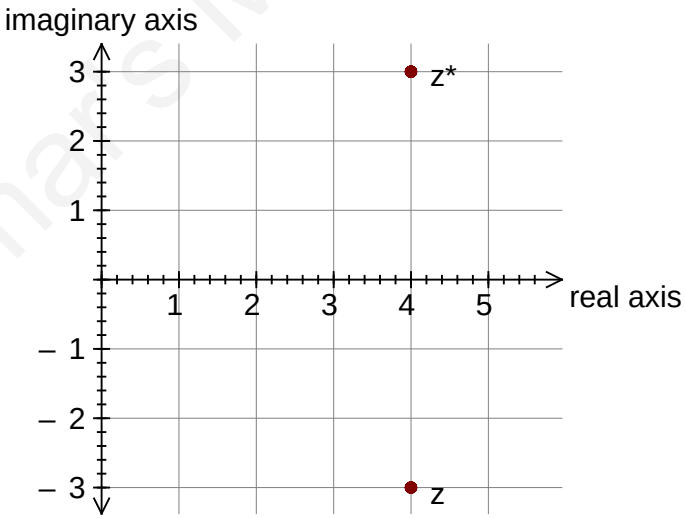
Complex numbers can be shown on an Argand diagram. The horizontal axis represents the real part of the complex number whilst the vertical axis represents the imaginary part.

Example:
Plot the complex numbers $z_1 = 4 + 2i$, $z_2 = 3 - i$, $z_3 = -2 + 3i$, $z_4 = -1 - 3i$ and $z_5 = i$ on an Argand diagram.



Example 2: Let $z = 3 - 4i$. Show z and z^* on an Argand diagram.

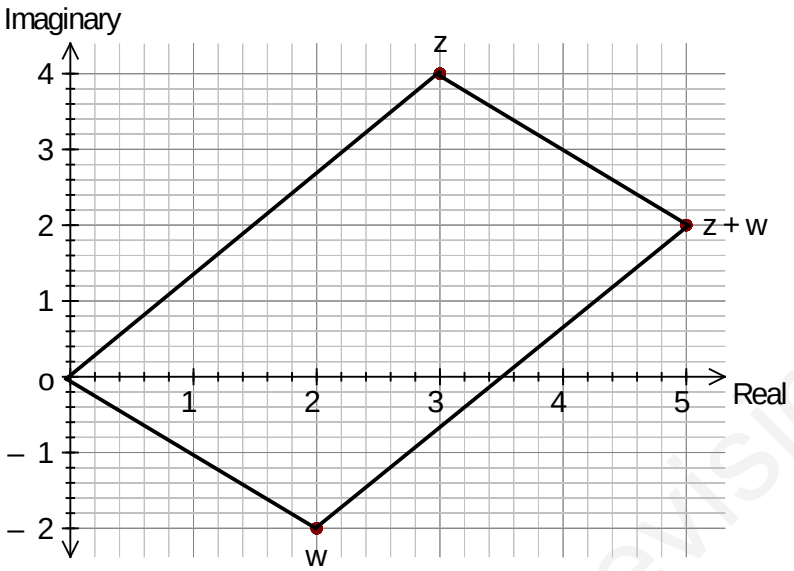
Solution: $z^* = 3 + 4i$.



Note: In an Argand diagram, z* is a reflection of z in the real axis.

Example 3: Let z = 3 + 4i and w = 2 – 2i. Show z, w and z + w on an Argand diagram.

Solution: z + w = 5 + 2i.

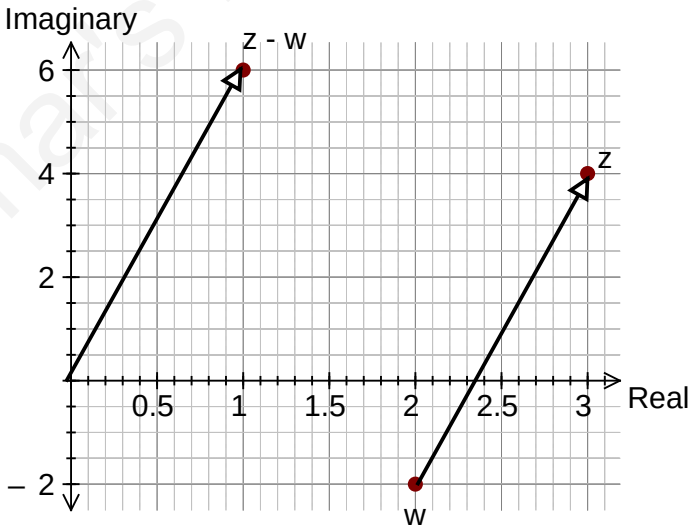


Notice that the points representing the origin, z, w and z + w form a parallelogram.

General result: If complex numbers z, w and z + w are represented by points Z, W and S in an Argand diagram, then OZSW is a parallelogram.

Example 4:
Let z = 3 + 4i and w = 2 – 2i. Show z, w and z - w on an Argand diagram.

Solution: z – w = 1 + 6i.



Notice this general result: If the complex numbers z and w are represented in an Argand diagram by points Z and W, then the translation which takes W to Z represents the complex number z – w.

Examination question:

Given that $z_1 = 1 + 2i$ and $z_2 = \frac{3}{5} + \frac{4}{5}i$, write z_1z_2 and $\frac{z_1}{z_2}$ in the form $p + iq$, where $p, q \in \mathbb{R}$.

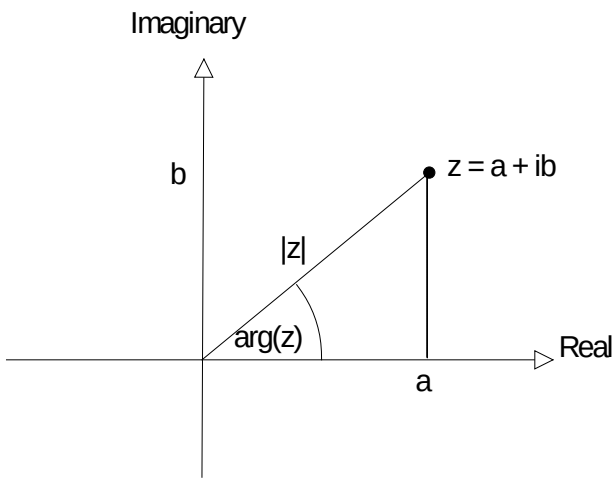
Plot the points representing z_1, z_2, z_1z_2 and $\frac{z_1}{z_2}$ on an Argand diagram.

In the diagram, the origin and the points representing $z_1z_2, \frac{z_1}{z_2}, z_3$ are the vertices of a rhombus.

Find z_3 and sketch the rhombus on this Argand diagram.

3.2 Modulus and argument of a complex number

Consider the complex number $z = a + ib$:



The **modulus** of z, written |z|, is the distance of z from the origin.

Therefore: $|z| = \sqrt{a^2 + b^2}$.

The **argument** of a complex number z is the angle that the line joining O to z makes with the positive real axis. Anticlockwise rotation is positive and clockwise rotation is negative. The argument is usually measured in radians and is chosen so that $-\pi < \arg(z) \leq \pi$.

Example: Find the modulus and argument of these complex numbers:

a) 2 + 7i
b) 5 – 2i
c) -4 + 3i
d) -2 – 3i
e) -5
f) 3i

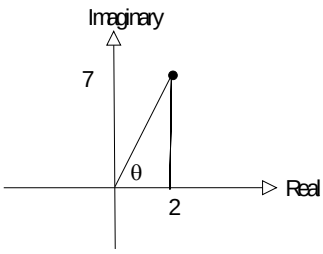
Solution: It helps to sketch an Argand diagram in each case:

a) $|2 + 7i| = \sqrt{2^2 + 7^2} = \sqrt{53}$

arg(2 + 7i) is angle θ in the diagram:

$$\tan\theta = \frac{7}{2}$$

i.e. $\theta = 1.29$ radians

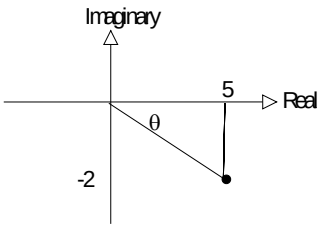


b) $|5 - 2i| = \sqrt{5^2 + (-2)^2} = \sqrt{29}$

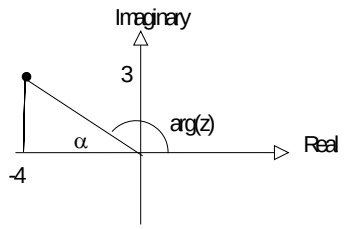
arg(5 - 2i) is shown by angle θ in the diagram.

$$\tan\theta = \frac{2}{5} \Rightarrow \theta = 0.381$$

i.e. $\theta = -0.381$ radians (it is negative as the number is below the axis).



c) $| - 4 + 3i | = \sqrt{(-4)^2 + 3^2} = 5$

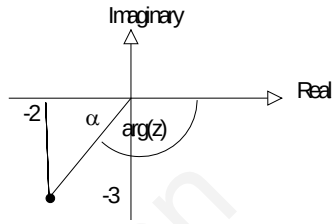


arg(-4 + 3i) is the angle shown on the diagram.
To find it, it is easiest to first find angle α:

$$\tan(\alpha) = \tfrac{3}{4} \qquad \Rightarrow \qquad \alpha = 0.644$$

Therefore, arg(-4 + 3i) = π – 0.644 = 2.50 radians.

d) $|-2 - 3i| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$

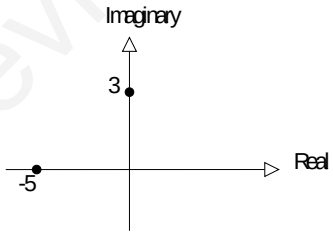


arg(-2 – 3i) is the angle shown on the diagram. As -2-3i is below the real axis, the argument will be negative.
It is simplest to initially find angle α:

$$\tan(\alpha) = \tfrac{3}{2} \qquad \Rightarrow \qquad \alpha = 0.983$$

π – 0.983 = 2.16 radians.
Therefore arg(-2 – 3i) = -2.16 radians.

e) |-5| = 5
arg(-5) = π rads



f) |3i| = 3
arg(3i) = ½ π rads

Section 4: Mixed complex numbers questions

Worked examination style question:

The complex number 2 + 5i is one root of the equation $x^2 + px + q = 0$, where p and q are real.

a) Write down the second root of this equation.
b) Find the values of p and q.

Solution:

a) As the quadratic equation has real coefficients, the second root must be the complex conjugate of the first, i.e. 2 – 5i.

b) We know that the sum of the roots is given by $-b/a = -p$.
But the sum of the roots is (2 + 5i) + (2 – 5i) = 4.
Therefore $-p = 4$ i.e. $p = -4$.

The product of the roots is $c/a = q$.
But the product of the roots is (2 + 5i)(2 – 5i) = 4 – 10i + 10i + 25 = 29.
So $q = 29$.

Examination question (OCR January 2005):

(i) The complex number z is such that $z^2 = 1 + i\sqrt{3}$. Find the two possible values of z in the form $a + ib$, where a and b are exact real numbers.
(ii) With the value of z from part (i) such that the real part of z is positive, show on an Argand diagram the points A and B representing z and z^2 respectively.

Examination question (OCR May 2005):

The complex numbers z_1 and z_2 are such that $z_1 = 1 - i$ and $z_2 = -\sqrt{3} + i$.

(i) Find $\frac{z_1}{z_2}$ in the form $x + iy$, where x and y are exact real numbers.

(i) Find the exact modulus and argument (in terms of π) of z_1 and z_2.

Self-review:

Syllabus	Self-review ☺ 😐 ☹
Understand the idea of a complex number, recall the meaning of the terms *real part*, *imaginary part*, *modulus*, *argument*, *conjugate*, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal;	
Carry out operations of *addition*, *subtraction*, *multiplication* and *division* of two complex numbers;	
Be able to use the result that, for a polynomial equation with real coefficients, any non-real roots occur in *conjugate pairs*;	
Represent complex numbers geometrically by means of an *Argand diagram*, and understand the geometrical effects of conjugating a complex number and of adding and subtracting two complex numbers;	
Find the two square roots of a complex number;	

Edexcel Past examination questions

1. Given that $z = 22 + 4\text{i}$ and $\frac{z}{w} = 6 - 8\text{i}$, find

(a) w in the form $a + b\text{i}$, where a and b are real,

(3)

(b) the argument of z, in radians to 2 decimal places.

(2)

[P4 January 2002 Qn 1]

2. Given that $2 + \text{i}$ is a root of the equation

$$z^2 + bz + c = 0, \text{ where } b \text{ and } c \text{ are real constants,}$$

(i) write down the other root of the equation,

(ii) find the value of b and the value of c.

(5)

[*P4 January 2002 Qn 5]

3. Given that $z = 3 + 4\text{i}$ and $w = -1 + 7\text{i}$,

(a) find $|w|$.

(1)

The complex numbers z and w are represented by the points A and B on an Argand diagram.

(b) Show points A and B on an Argand diagram.

(1)

(c) Prove that $\triangle OAB$ is an isosceles right-angled triangle.

(5)

(d) Find the exact value of $\arg\left(\frac{z}{w}\right)$.

(3)

[P4 June 2002 Qn 5]

4. Given that $z = 3 - 3\text{i}$ express, in the form $a + \text{i}b$, where a and b are real numbers,

(a) z^2,

(2)

(b) $\frac{1}{z}$.

(2)

(c) Find the exact value of each of $|z|, |z^2|$ and $\left|\frac{1}{z}\right|$.

(2)

The complex numbers z, z^2 and $\frac{1}{z}$ are represented by the points A, B and C respectively on an Argand diagram. The real number 1 is represented by the point D, and O is the origin.

(d) Show the points A, B, C and D on an Argand diagram.

(2)

(e) Prove that $\triangle OAB$ is similar to $\triangle OCD$.

(3)

[P4 January 2003 Qn 6]

5. (a) Using that 3 is the real root of the cubic equation $x^3 - 27 = 0$, show that the complex roots of the cubic satisfy the quadratic equation $x^2 + 3x + 9 = 0$.

(2)

(b) Hence, or otherwise, find the three cube roots of 27, giving your answers in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{R}$.

(3)

(c) Show these roots on an Argand diagram.

(2)

[P4 June 2003 Qn 3]

6.

$$z = \frac{a + 3\mathrm{i}}{2 + a\mathrm{i}}, \qquad a \in \mathbb{R}.$$

(a) Given that $a = 4$, find $|z|$.

(3)

(b) Show that there is only one value of a for which $\arg z = \frac{\pi}{4}$, and find this value.

(6)

[P4 June 2003 Qn 5]

7. Given that $z = 2 - 2\mathrm{i}$ and $w = -\sqrt{3} + \mathrm{i}$,

(a) find the modulus and argument of wz^2.

(6)

(b) Show on an Argand diagram the points A, B and C which represent z, w and wz^2 respectively, and determine the size of angle BOC.

(4)

[P4 January 2004 Qn 3]

8. The complex number $z = a + \mathrm{i}b$, where a and b are real numbers, satisfies the equation

$$z^2 + 16 - 30\mathrm{i} = 0.$$

(a) Show that $ab = 15$.

(2)

(b) Write down a second equation in a and b and hence find the roots of

$$z^2 + 16 - 30\mathrm{i} = 0.$$

(4)

[P4 June 2004 Qn 3]

9. Given that $z = 1 + \sqrt{3}\mathrm{i}$ and that $\frac{w}{z} = 2 + 2\mathrm{i}$, find

(*a*) w in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{R}$,

(3)

(*b*) the argument of w,

(2)

(*c*) the exact value for the modulus of w.

(2)

On an Argand diagram, the point A represents z and the point B represents w.

(*d*) Draw the Argand diagram, showing the points A and B.

(2)

(*e*) Find the distance AB, giving your answer as a simplified surd.

(2)

[P4 June 2004 Qn 5]

10. Given that $z = -2\sqrt{2} + 2\sqrt{2}\mathrm{i}$ and $w = 1 - \mathrm{i}\sqrt{3}$, find

(*a*) $\left|\frac{z}{w}\right|$,

(3)

(*b*) $\arg\left(\frac{z}{w}\right)$.

(3)

(*c*) On an Argand diagram, plot points A, B, C and D representing the complex numbers z, w, $\left(\frac{z}{w}\right)$ and 4, respectively.

(3)

(*d*) Show that $\angle AOC = \angle DOB$.

(2)

(*e*) Find the area of triangle AOC.

(2)

[FP1/P4 January 2005 Qn 8]

11. Given that -2 is a root of the equation $z^3 + 6z + 20 = 0$,

(*a*) find the other two roots of the equation,

(3)

(*b*) show, on a single Argand diagram, the three points representing the roots of the equation,

(1)

(*c*) prove that these three points are the vertices of a right-angled triangle.

(2)

[FP1/P4 June 2005 Qn 2]

12. $z = -4 + 6\text{i}$.

(*a*) Calculate arg z, giving your answer in radians to 3 decimal places.

(2)

The complex number w is given by $w = \dfrac{A}{2-\text{i}}$, where A is a positive constant. Given that $|w| = \sqrt{20}$,

(*b*) find w in the form $a + \text{i}b$, where a and b are constants,

(4)

(*c*) calculate arg $\dfrac{w}{z}$.

(3)

[FP1/P4 June 2005 Qn 5]

13. Given that $\dfrac{z+2\text{i}}{z-\lambda\text{i}} = \text{i}$, where λ is a positive, real constant,

(*a*) show that $z = \left(\dfrac{\lambda}{2}+1\right) + \text{i}\left(\dfrac{\lambda}{2}-1\right)$.

(5)

Given also that arg z = arctan $\frac{1}{2}$, calculate

(*b*) the value of λ,

(3)

(*c*) the value of $|z|^2$.

(2)

[FP1/P4 January 2006 Qn 3]

14. The complex numbers z and w satisfy the simultaneous equations

$$2z + \text{i}w = -1,$$

$$z - w = 3 + 3\text{i}.$$

(*a*) Use algebra to find z, giving your answers in the form $a + \text{i}b$, where a and b are real.

(4)

(*b*) Calculate arg z, giving your answer in radians to 2 decimal places.

(2)

[FP1 June 2006 Qn 1]

15. (*a*) Find the roots of the equation

$$z^2 + 2z + 17 = 0,$$

giving your answers in the form $a + \text{i}b$, where a and b are integers.

(3)

(*b*) Show these roots on an Argand diagram.

(1)

[FP1 January 2007 Qn 1]

16. The complex numbers z_1 and z_2 are given by

$$z_1 = 5 + 3\text{i},$$
$$z_1 = 1 + p\text{i},$$

where p is an integer.

(*a*) Find $\frac{z_2}{z_1}$, in the form $a + \text{i}b$, where a and b are expressed in terms of p.

(3)

Given that $\arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{4}$,

(*b*) find the value of p.

(2)

[FP1 January 2007 Qn 3]

17.

$$z = \sqrt{3} - \text{i}.$$

z^* is the complex conjugate of z.

(*a*) Show that $\frac{z}{z^*} = \frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}$.

(3)

(*b*) Find the value of $\left|\frac{z}{z^*}\right|$.

(2)

(*c*) Verify, for $z = \sqrt{3} - \text{i}$, that $\arg\frac{z}{z^*} = \arg z - \arg z^*$.

(4)

(*d*) Display z, z^* and $\frac{z}{z^*}$ on a single Argand diagram.

(2)

(*e*) Find a quadratic equation with roots z and z^* in the form $ax^2 + bx + c = 0$, where a, b and c are real constants to be found.

(2)

[FP1 June 2007 Qn 6]

18. Given that $x = -\frac{1}{2}$ is the real solution of the equation

$$2x^3 - 11x^2 + 14x + 10 = 0,$$

find the two complex solutions of this equation.

(6)

[FP1 January 2008 Qn 2]

19. $z = -2 + \mathrm{i}$.

(*a*) Express in the form $a + \mathrm{i}b$

(i) $\dfrac{1}{z}$

(ii) z^2.

(4)

(*b*) Show that $|z^2 - z| = 5\sqrt{2}$.

(2)

(*c*) Find $\arg(z^2 - z)$.

(2)

(*d*) Display z and $z^2 - z$ on a single Argand diagram.

(2)

[FP1 January 2008 Qn 6]

20. (*a*) Write down the value of the real root of the equation

$$x^3 - 64 = 0.$$

(1)

(*b*) Find the complex roots of $x^3 - 64 = 0$, giving your answers in the form $a + \mathrm{i}b$, where a and b are real.

(4)

(*c*) Show the three roots of $x^3 - 64 = 0$ on an Argand diagram.

(2)

[FP1 June 2008 Qn 1]

21. The complex number z is defined by

$$z = \frac{a + 2\mathrm{i}}{a - \mathrm{i}}, \qquad a \in \mathbb{R}, \quad a > 0 .$$

Given that the real part of z is $\frac{1}{2}$, find

(*a*) the value of a,

(4)

(*b*) the argument of z, giving your answer in radians to 2 decimal places.

(3)

[FP1 June 2008 Qn 3]

22. $$\mathrm{f}(x) = 2x^3 - 8x^2 + 7x - 3.$$

Given that $x = 3$ is a solution of the equation $\mathrm{f}(x) = 0$, solve $\mathrm{f}(x) = 0$ completely.

(5)

[FP1 Jan 2009 Qn 1]

23. Given that $z_1 = 3 + 2\text{i}$ and $z_2 = \dfrac{12 - 5\text{i}}{z_1}$,

(*a*) find z_2 in the form $a + \text{i}b$, where a and b are real.

(2)

(*b*) Show, on an Argand diagram, the point P representing z_1 and the point Q representing z_2.

(2)

(*c*) Given that O is the origin, show that $\angle POQ = \dfrac{\pi}{2}$.

(2)

The circle passing through the points O, P and Q has centre C. Find

(*d*) the complex number represented by C,

(2)

(*e*) the exact value of the radius of the circle.

(2)

[FP1 Jan 2009 Qn 9]

24. The complex numbers z_1 and z_2 are given by

$$z_1 = 2 - \text{i} \text{ and } z_2 = -8 + 9\text{i}$$

(*a*) Show z_1 and z_2 on a single Argand diagram.

(1)

Find, showing your working,

(*b*) the value of $|z_1|$,

(2)

(*c*) the value of arg z_1, giving your answer in radians to 2 decimal places,

(2)

(*d*) $\dfrac{z_2}{z_1}$ in the form $a + b\text{i}$, where a and b are real.

(3)

[FP1 June 2009 Qn 1]

25.

$$\text{f}(x) = (x^2 + 4)(x^2 + 8x + 25)$$

(*a*) Find the four roots of $\text{f}(x) = 0$.

(5)

(*b*) Find the sum of these four roots.

(2)

[FP1 June 2009 Qn 3]

26. The complex numbers z_1 and z_2 are given by

$$z_1 = 2 + 8\text{i} \quad \text{and} \quad z_2 = 1 - \text{i}$$

Find, showing your working,

(*a*) $\dfrac{z_1}{z_2}$ in the form $a + b\text{i}$, where a and b are real,

(3)

(*b*) the value of $\left|\dfrac{z_1}{z_2}\right|$,

(2)

(*c*) the value of $\arg \dfrac{z_1}{z_2}$, giving your answer in radians to 2 decimal places.

(2)

[FP1 Jan 2010 Qn 1]

27. Given that 2 and 5 + 2i are roots of the equation

$$x^3 - 12x^2 + cx + d = 0, \quad c, d \in \mathbb{R},$$

(*a*) write down the other complex root of the equation.

(1)

(*b*) Find the value of c and the value of d.

(5)

(*c*) Show the three roots of this equation on a single Argand diagram.

(2)

[FP1 Jan 2010 Qn 6]

28.

$$z = 2 - 3\text{i}$$

(*a*) Show that $z^2 = -5 - 12\text{i}$.

(2)

Find, showing your working,

(*b*) the value of $|z^2|$,

(2)

(*c*) the value of $\arg(z^2)$, giving your answer in radians to 2 decimal places.

(2)

(*d*) Show z and z^2 on a single Argand diagram.

(1)

[FP1 June 2010 Qn 1]

29. $$f(x) = x^3 + x^2 + 44x + 150.$$

Given that $f(x) = (x + 3)(x^2 + ax + b)$, where a and b are real constants,

(a) find the value of a and the value of b.

(2)

(b) Find the three roots of $f(x) = 0$.

(4)

(c) Find the sum of the three roots of $f(x) = 0$.

(1)

[FP1 June 2010 Qn 4]

30. $$z = 5 - 3i, \quad w = 2 + 2i$$

Express in the form $a + bi$, where a and b are real constants,

(a) z^2,

(2)

(b) $\dfrac{z}{w}$.

(3)

[FP1 Jan 2011 Qn 1]

31. Given that $2 - 4i$ is a root of the equation

$$z^2 + pz + q = 0,$$

where p and q are real constants,

(a) write down the other root of the equation,

(1)

(b) find the value of p and the value of q.

(3)

[FP1 Jan 2011 Qn 4]

32. $$z = -24 - 7i$$

(a) Show z on an Argand diagram.

(1)

(b) Calculate arg z, giving your answer in radians to 2 decimal places.

(2)

It is given that

$$w = a + bi, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}.$$

Given also that $|w| = 4$ and $\arg w = \frac{5\pi}{6}$,

(*c*) find the values of *a* and *b*,

(3)

(*d*) find the value of $|zw|$.

(3)

[FP1 Jan 2011 Qn 7]

33.

$$z_1 = -2 + \mathrm{i}$$

(*a*) Find the modulus of z_1.

(1)

(*b*) Find, in radians, the argument of z_1, giving your answer to 2 decimal places.

(2)

The solutions to the quadratic equation

$$z^2 - 10z + 28 = 0$$

are z_2 and z_3.

(*c*) Find z_2 and z_3, giving your answers in the form $p \pm \mathrm{i}\sqrt{q}$, where *p* and *q* are integers.

(3)

(*d*) Show, on an Argand diagram, the points representing your complex numbers z_1, z_2 and z_3.

(2)

[FP1 June 2011 Qn 2]

34. Given that $z = x + \mathrm{i}y$, find the value of *x* and the value of *y* such that

$$z + 3\mathrm{i}z^* = -1 + 13\mathrm{i}$$

where z^* is the complex conjugate of *z*.

(7)

[FP1 June 2011 Qn 6]

35. Given that $z_1 = 1 - \mathrm{i}$,

(*a*) find $\arg(z_1)$.

(2)

Given also that $z_2 = 3 + 4\mathrm{i}$, find, in the form $a + \mathrm{i}b$, $a, b \in \mathbb{R}$,

(*b*) $z_1 z_2$,

(2)

(*c*) $\frac{z_2}{z_1}$.

(3)

In part (*b*) and part (*c*) you must show all your working clearly.

[FP1 Jan 2012 Qn 1]

36. The roots of the equation

$$z^3 - 8z^2 + 22z - 20 = 0$$

are z_1, z_2 and z_3 .

(*a*) Given that $z_1 = 3 + \mathrm{i}$, find z_2 and z_3.

(4)

(*b*) Show, on a single Argand diagram, the points representing z_1, z_2 and z_3.

(2)

[FP1 Jan 2012 Qn 5]

37.

$$\mathrm{f}(x) = 2x^3 - 6x^2 - 7x - 4.$$

(*a*) Show that f(4) = 0.

(1)

(*b*) Use algebra to solve f(*x*) = 0 completely.

(4)

[FP1 June 2012 Qn 1]

38.

$$z = 2 - \mathrm{i}\sqrt{3}.$$

(*a*) Calculate arg *z*, giving your answer in radians to 2 decimal places.

(2)

Use algebra to express

(*b*) $z + z^2$ in the form $a + b\mathrm{i}\sqrt{3}$, where *a* and *b* are integers,

(3)

(*c*) $\dfrac{z+7}{z-1}$ in the form $c + d\mathrm{i}\sqrt{3}$, where *c* and *d* are integers.

(4)

Given that

$$w = \lambda - 3\mathrm{i},$$

where λ is a real constant, and $\arg(4 - 5\mathrm{i} + 3w) = -\dfrac{\pi}{2}$,

(*d*) find the value of λ.

(2)

[FP1 June 2012 Qn 7]

39.

$$z = \frac{50}{3+4\mathrm{i}}.$$

Find, in the form a + i*b* where $a, b \in \mathbb{R}$,

(*a*) *z*,

(2)

(*b*) z^2.

(2)

Find

(*c*) $|z|$, **(2)**

(*d*) arg z^2, giving your answer in degrees to 1 decimal place. **(2)**

[FP1 Jan 2013 Qn 2]

40. $$f(x) = (4x^2 + 9)(x^2 - 6x + 34).$$

(*a*) Find the four roots of f (x) = 0.

Give your answers in the form $x = p + iq$, where p and q are real.

(5)

(*b*) Show these four roots on a single Argand diagram.

(2)

[FP1 Jan 2013 Qn 5]

41. Given that $x = \frac{1}{2}$ is a root of the equation

$$2x^3 - 9x^2 + kx - 13 = 0, \qquad k \in \mathbb{R}$$

find

(*a*) the value of k,

(3)

(*b*) the other 2 roots of the equation.

(4)

[FP1 June 2013 Qn 3]

42. $$z_1 = 2 + 3i, \qquad z_2 = 3 + 2i, \qquad z_3 = a + bi, \qquad a, b \in \mathbb{R}$$

(*a*) Find the exact value of $|z_1 + z_2|$.

(2)

Given that $w = \frac{z_1 z_3}{z_2}$,

(*b*) find w in terms of a and b, giving your answer in the form $x + iy$, $\quad x, y \in \mathbb{R}$.

(4)

Given also that $w = \frac{17}{13} - \frac{7}{13}i$,

(*c*) find the value of a and the value of b,

(3)

(*d*) find arg w, giving your answer in radians to 3 decimal places.

(2)

[FP1 June 2013 Qn 7]

43. The complex numbers z and w are given by

$$z = 8 + 3i, \qquad w = -2i$$

Express in the form $a + bi$, where a and b are real constants,

(*a*) $z - w$,

(1)

(*b*) zw.

(2)

[FP1 June 2013_R Qn 1]

44. $$f(x) = (4x^2 + 9)(x^2 - 2x + 5)$$

(*a*) Find the four roots of $f(x) = 0$.

(4)

(*b*) Show the four roots of $f(x) = 0$ on a single Argand diagram.

(2)

[FP1 June 2013_R Qn 4]

45. The complex number w is given by

$$w = 10 - 5\text{i}$$

(*a*) Find $|w|$.

(1)

(*b*) Find arg w, giving your answer in radians to 2 decimal places

(2)

The complex numbers z and w satisfy the equation

$$(2 + \text{i})(z + 3\text{i}) = w$$

(*c*) Use algebra to find z, giving your answer in the form $a + b\text{i}$, where a and b are real numbers.

(4)

Given that

$$\arg(\lambda + 9\text{i} + w) = \frac{\pi}{4}$$

where λ is a real constant,

(*d*) find the value of λ.

(2)

[FP1 June 2013_R Qn 9]

46. The complex numbers z_1 and z_2 are given by

$$z_1 = p + 2\text{i} \text{ and } z_2 = 1 - 2\text{i}$$

where p is an integer.

(*a*) Find $\frac{z_1}{z_2}$ in the form $a + b\text{i}$ where a and b are real. Give your answer in its simplest form in terms of p.

(4)

Given that $\left|\frac{z_1}{z_2}\right| = 13$,

(*b*) find the possible values of p.

(4)

[FP1 June 2014 Qn 1]

47. Given that 2 and 1 – 5i are roots of the equation

$$x^3 + px^2 + 30x + q = 0, \qquad p, q \in \mathbb{R}$$

(*a*) write down the third root of the equation.

(1)

(*b*) Find the value of p and the value of q.

(5)

(*c*) Show the three roots of this equation on a single Argand diagram.

(2)

[FP1 June 2014 Qn 3]

48. The roots of the equation

$$2z^3 - 3z^2 + 8z + 5 = 0$$

are z_1, z_2 and z_3.

Given that $z_1 = 1 + 2\text{i}$, find z_2 and z_3.

(5)

[FP1 June 2014_R Qn 1]

49. The complex number z is given by

$$z = \frac{p + 2\text{i}}{3 + p\text{i}}$$

where p is an integer.

(*a*) Express z in the form $a + b\text{i}$ where a and b are real. Give your answer in its simplest form in terms of p.

(4)

(*b*) Given that $\arg(z) = \theta$, where $\tan\theta = 1$ find the possible values of p.

(5)

[FP1 June 2014_R Qn 4]