OCR Further Pure 1

Past paper questions
Proof by Induction

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Complex Numbers

- Complex numbers start with one idea only; that we can find a number that squares to -1; we call it i. Therefore $i^2 = -1$. It is not a number that exists on the number line so it is refered to as *complex* or *imaginary*. Therefore the square root of any negative number can now be calculated; $\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i$.

- In general a complex number can consist of a real part and an imaginary/complex part i.e. $a+ib$, where a is the real part and b is the complex part. We write $\text{Re}(a+ib) = a$ and $\text{Im}(a+ib) = b$. It is important to note that a and b themselves *must* be real numbers.

- We can use complex numbers to solve *any* quadratic equation. For example solve $3x^2 + 2x + 7 = 0$ by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \times 3 \times 7}}{2 \times 3} = -\frac{1}{3} \pm i\frac{2\sqrt{5}}{3}.$$

- A complex (or real) number can be represented as a point in an *Argand* diagram. So the complex number $6 + 2i$ would be the point 6 across and 2 up, at the equivalent point where $(6, 2)$ would be in a Cartesian coordinate system.

- The complex conjugate $(z*)$ of a complex number (z) is where the complex part has the sign changed. For example if $z = 3 - 7i$, then $z* = 3 + 7i$. Real numbers are, therefore, their own conjugates. Any number with an imaginary component is reflected in the real axis in the Argand diagram.

- If two complex numbers are equal, then the real parts must be equal and the complex parts must be equal: i.e.

$$\begin{aligned} z_1 = z_2 \quad &\Rightarrow \quad \text{Re}(z_1) = \text{Re}(z_2) \quad \text{and} \quad \text{Im}(z_1) = \text{Im}(z_2), \\ a + ib = c + id \quad &\Rightarrow \quad a = c \quad \text{and} \quad b = d. \end{aligned}$$

- To add, subtract or multiply complex numbers the results are pretty obvious:

$$\begin{aligned} (a+bi) + (c+id) &= (a+c) + i(b+d) \\ (a+bi) - (c+id) &= (a-c) + i(b-d) \\ (a+bi)(c+id) &= ac + adi + bci + bdi^2 = (ac-bd) + i(ad+bc) \end{aligned}$$

- To divide by a complex number use a trick taken from surds; in C1 if the bottom line was $a \pm b\sqrt{k}$ then you multiplied top and bottom by $a \mp b\sqrt{k}$. In FP1 if you want to divide by $a \pm ib$, then you multiply top and bottom by the complex conjugate $a \mp ib$. For example:

$$\frac{3-2i}{2+5i} = \frac{3-2i}{2+5i} \times \frac{2-5i}{2-5i} = \frac{6+10i^2-4i-15i}{4-25i^2+10i-10i} = \frac{-4-19i}{29}.$$

- Complex numbers exhibit the elegant property of *closure*[2]. This means that any operation on complex numbers involving $+, -, \times, \div, \sqrt{\ldots}, \sqrt[n]{\ldots}$ etc. will produce an answer that is also complex[3]. This allows us to state that the answer to a given problem *must* be $a + ib$ for some a and b and then proceed to calculate a and b by equating the real part and, separately, the imaginary part.

- In the above problem to find $\frac{3-2i}{2+5i}$ we could also have approached it by stating that the answer is $a+ib$ and manipulating:

$$\begin{aligned}\frac{3-2i}{2+5i} &= a+ib \\ 3-2i &= (a+ib)(2+5i) \\ 3-2i &= (2a-5b)+i(5a+2b).\end{aligned}$$

  This yields the simultaneous equations $3=2a-5b$ and $-2=5a+2b$. These solve to $a=-\frac{4}{29}$ and $b=-\frac{19}{29} \Rightarrow \frac{-4-19i}{29}$, just as before. I wouldn't use this method in this case but I would certainly use it...

- ...to find square roots. The square roots of 16 are (obviously) ± 4. With the exception of zero, we should expect two roots and the same is true of complex numbers. For example find the square roots of $8-6i$: We know that the answers must be of the form $a+ib$ such that

$$\begin{aligned}8-6i &= (a+ib)^2 \\ 8-6i &= (a^2-b^2)+(2ab)i \\ \text{Therefore, } 8 &= a^2-b^2 \text{ and } -6=2ab.\end{aligned}$$

  From the second we find $b=-\frac{3}{a}$. Putting this in the first we find $0=a^4-8a^2-9=(a^2-9)(a^2+1)$. The first bracket yields $a=\pm 3$. (The second bracket yields $a=\pm i$, but we can discard this because a must be real.) Therefore this yields the square roots $3-i$ and $-3+i$. In the Argand diagram you should find that square roots come out in opposite directions from the origin.

- If a polynomial has *real coefficients* then its roots are either real, or exist in complex conjugate pairs. Therefore if $z=a\pm ib$ is a root, then so is $z=a\mp ib$.

  For example, given that $z^4-z^3+2z^2+7z-5=0$ has one root $1-2i$, solve the equation fully. Since the coefficients are real we know that the conjugate $1+2i$ must also be a root. Therefore $(z-(1-2i))$ and $(z-(1+2i))$ must be factors by the factor theorem. Multiplying out the two factors we find $(z-(1-2i))(z-(1+2i))=(z^2-2z+5)$ which must also be a factor. By polynomial division we find

$$z^4-z^3+2z^2+7z-5=(z^2-2z+5)(z^2+z-1)=0.$$

  The second quadratic solves to $z=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}$. Therefore the solutions are

$$z=1-2i, \quad z=1+2i, \quad z=-\tfrac{1}{2}+\tfrac{\sqrt{5}}{2}, \quad z=-\tfrac{1}{2}-\tfrac{\sqrt{5}}{2}.$$

- The modulus of a complex number ($z=x+iy$) is defined $|z|=\sqrt{x^2+y^2}$. It represents the distance of a complex number from the origin. For example the modulus of $z=2-2\sqrt{3}i$ would be $|z|=\sqrt{2^2+(-2\sqrt{3})^2}=4$.

- The argument of a complex number is defined as the angle a line from the origin to a complex number makes with the positive real axis. By convention $-\pi<\arg(z)\leqslant\pi$. For example

$$\arg(4) = 0,$$
$$\arg(i) = \frac{\pi}{2},$$
$$\arg(-3) = \pi,$$
$$\arg(1+i) = \frac{\pi}{4},$$
$$\arg(-1-i) = -\frac{3\pi}{4}.$$

Arguments are best calculated by drawing a suitable right angled triangle in an Argand diagram and then calculating the desired angle (not always an angle in the triangle you've drawn, but π minus it, etc.)

- You must be able to sketch loci of points obeying a rule defined by a modulus or an argument. The most important fact here is often that the operation *subtraction* takes you *from* one complex number *to* another[4]; i.e. $z - w$ takes you *from* w *to* z. An addition can be converted into a subtraction by $z + w = z - (-w)$; this therefore represents the movement from $-w$ to z.
- This idea allows us to draw certain loci very easily indeed.

  For example: $|z| = 4$ means the length of z from the origin is 4; i.e. a circle of radius 4, centre the origin.

  For example: $|z - 2| = 5$ means the length travelling from 2 to z is 5, so a circle radius 5, centre 2(+0i).

  For example: $|z + i| < 2$ is the same as $|z - (-i)| < 2$ which means the length travelling from $-i$ to z is less than 2, so the inside of a circle radius 2, centre $(0) - i$.

  For example: $|z| = |z + 1 - i|$ is the same as $|z| = |z - (-1 + i)|$ which means the length travelling from 0 to z must be the same as the distance travelling from $-1 + i$ to z so it must be the perpendicular bisector of 0 and $-1 + i$, i.e. $y = x + 1$.
- The above type of questions can also be done using a method in your textbook (see top of P144, "Method 2"), but I prefer the 'intuitive' way demonstrated above.
- Argument loci also come up and we can use the same principles.

  For example: $\arg(z) = \frac{\pi}{2}$ means the argument z makes is $\frac{\pi}{2}$ so it is a vertical line going up from the origin (with a hollow circle drawn at the origin to indicate that it is not included in the half-line).

  For example: $\arg(z - i) = \frac{\pi}{6}$ means the argument going from i to z is $\frac{\pi}{6}$ so it is a half-line from i (hollow circle) at angle $\frac{\pi}{6}$ with the positive real axis.

1.

The complex numbers $2 + 3\text{i}$ and $4 - \text{i}$ are denoted by z and w respectively. Express each of the following in the form $x + \text{i}y$, showing clearly how you obtain your answers.

(i) $z + 5w$, [2]

(ii) z^*w, where z^* is the complex conjugate of z, [3]

(iii) $\frac{1}{w}$. [2]

(Q3, June 2005)

2.

Use an algebraic method to find the square roots of the complex number $21 - 20\text{i}$. [6]

(Q4, June 2005)

3.

The loci C_1 and C_2 are given by

$$|z - 2\text{i}| = 2 \qquad \text{and} \qquad |z + 1| = |z + \text{i}|$$

respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. [5]

(ii) Hence write down the complex numbers represented by the points of intersection of C_1 and C_2. [2]

(Q6, June 2005)

4.

(i) Express $(1 + 8\text{i})(2 - \text{i})$ in the form $x + \text{i}y$, showing clearly how you obtain your answer. [2]

(ii) Hence express $\frac{1 + 8\text{i}}{2 + \text{i}}$ in the form $x + \text{i}y$. [3]

(Q1, Jan 2006)

5.

(a) The complex number $3 + 2\text{i}$ is denoted by w and the complex conjugate of w is denoted by w^*. Find

(i) the modulus of w, [1]

(ii) the argument of w^*, giving your answer in radians, correct to 2 decimal places. [3]

(b) Find the complex number u given that $u + 2u^* = 3 + 2\text{i}$. [4]

(c) Sketch, on an Argand diagram, the locus given by $|z + 1| = |z|$. [2]

(Q7, Jan 2006)

6.

The complex numbers $3 - 2\text{i}$ and $2 + \text{i}$ are denoted by z and w respectively. Find, giving your answers in the form $x + \text{i}y$ and showing clearly how you obtain these answers,

(i) $2z - 3w$, [2]

(ii) $(\text{i}z)^2$, [3]

(iii) $\dfrac{z}{w}$. [3]

(Q5, June 2006)

7.

In an Argand diagram the loci C_1 and C_2 are given by

$$|z| = 2 \qquad \text{and} \qquad \arg z = \tfrac{1}{3}\pi$$

respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. [5]

(ii) Hence find, in the form $x + \text{i}y$, the complex number representing the point of intersection of C_1 and C_2. [2]

(Q6, June 2006)

8.

Use an algebraic method to find the square roots of the complex number $15 + 8\text{i}$. [6]

(Q2, Jan 2007)

9.

(i) Sketch, on an Argand diagram, the locus given by $|z - 1 + \mathrm{i}| = \sqrt{2}$. [3]

(ii) Shade on your diagram the region given by $1 \leqslant |z - 1 + \mathrm{i}| \leqslant \sqrt{2}$. [3]

(Q4, Jan 2007)

10.

(i) Verify that $z^3 - 8 = (z - 2)(z^2 + 2z + 4)$. [1]

(ii) Solve the quadratic equation $z^2 + 2z + 4 = 0$, giving your answers exactly in the form $x + \mathrm{i}y$. Show clearly how you obtain your answers. [3]

(iii) Show on an Argand diagram the roots of the cubic equation $z^3 - 8 = 0$. [3]

(Q5, Jan 2007)

11.

The complex number $a + \mathrm{i}b$ is denoted by z. Given that $|z| = 4$ and $\arg z = \frac{1}{3}\pi$, find a and b. [4]

(Q1, June 2007)

12.

The loci C_1 and C_2 are given by $|z - 3| = 3$ and $\arg(z - 1) = \frac{1}{4}\pi$ respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. [6]

(ii) Indicate, by shading, the region of the Argand diagram for which

$$|z - 3| \leqslant 3 \quad \text{and} \quad 0 \leqslant \arg(z - 1) \leqslant \tfrac{1}{4}\pi.$$ [2]

(Q8, June 2007)

13.

(i) Use an algebraic method to find the square roots of the complex number $16 + 30\mathrm{i}$. [6]

(ii) Use your answers to part **(i)** to solve the equation $z^2 - 2z - (15 + 30\mathrm{i}) = 0$, giving your answers in the form $x + \mathrm{i}y$. [5]

(Q10, June 2007)

14.

The complex number $3 - 4\text{i}$ is denoted by z. Giving your answers in the form $x + \text{i}y$, and showing clearly how you obtain them, find

(i) $2z + 5z^*$, [2]

(ii) $(z - \text{i})^2$, [3]

(iii) $\frac{3}{z}$. [3]

(Q4, Jan 2008)

15.

The loci C_1 and C_2 are given by

$$|z| = |z - 4\text{i}| \qquad \text{and} \qquad \arg z = \tfrac{1}{6}\pi$$

respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. [5]

(ii) Hence find, in the form $x + \text{i}y$, the complex number represented by the point of intersection of C_1 and C_2. [3]

(Q6, Jan 2008)

16.

The complex number $3 + 4\text{i}$ is denoted by a.

(i) Find $|a|$ and $\arg a$. [2]

(ii) Sketch on a single Argand diagram the loci given by

(a) $|z - a| = |a|$, [2]

(b) $\arg(z - 3) = \arg a$. [3]

(Q2, June 2008)

17.

(i) Use an algebraic method to find the square roots of the complex number $5 + 12\text{i}$. [5]

(ii) Find $(3 - 2\text{i})^2$. [2]

(iii) Hence solve the quartic equation $x^4 - 10x^2 + 169 = 0$. [4]

(Q9, June 2008)

18.

Express $\dfrac{2+3\mathrm{i}}{5-\mathrm{i}}$ in the form $x + \mathrm{i}y$, showing clearly how you obtain your answer. **[4]**

(Q1, Jan 2009)

19.

(i) Use an algebraic method to find the square roots of the complex number $2 + \mathrm{i}\sqrt{5}$. Give your answers in the form $x + \mathrm{i}y$, where x and y are exact real numbers. **[6]**

(ii) Hence find, in the form $x + \mathrm{i}y$ where x and y are exact real numbers, the roots of the equation

$$z^4 - 4z^2 + 9 = 0.$$ **[4]**

(iii) Show, on an Argand diagram, the roots of the equation in part **(ii)**. **[1]**

(iv) Given that α is the root of the equation in part **(ii)** such that $0 < \arg \alpha < \frac{1}{2}\pi$, sketch on the same Argand diagram the locus given by $|z - \alpha| = |z|$. **[3]**

(Q10, Jan 2009)

20.

The complex numbers z and w are given by $z = 5 - 2\mathrm{i}$ and $w = 3 + 7\mathrm{i}$. Giving your answers in the form $x + \mathrm{i}y$ and showing clearly how you obtain them, find

(i) $4z - 3w$, **[2]**

(ii) z^*w. **[2]**

(Q3, June 2009)

21.

The complex number $3 - 3\mathrm{i}$ is denoted by a.

(i) Find $|a|$ and $\arg a$. **[2]**

(ii) Sketch on a single Argand diagram the loci given by

(a) $|z - a| = 3\sqrt{2}$, **[3]**

(b) $\arg(z - a) = \frac{1}{4}\pi$. **[3]**

(iii) Indicate, by shading, the region of the Argand diagram for which

$$|z - a| \geqslant 3\sqrt{2} \quad \text{and} \quad 0 \leqslant \arg(z - a) \leqslant \tfrac{1}{4}\pi.$$ **[3]**

(Q6, June 2009)

22.

The complex number z satisfies the equation $z + 2iz^* = 12 + 9i$. Find z, giving your answer in the form $x + iy$. **[5]**

(Q3, Jan 2010)

23.

The complex number a is such that $a^2 = 5 - 12i$.

(i) Use an algebraic method to find the two possible values of a. **[5]**

(ii) Sketch on a single Argand diagram the two possible loci given by $|z - a| = |a|$. **[4]**

(Q8, Jan 2010)

24.

The complex numbers a and b are given by $a = 7 + 6i$ and $b = 1 - 3i$. Showing clearly how you obtain your answers, find

(i) $|a - 2b|$ and $\arg(a - 2b)$, **[4]**

(ii) $\dfrac{b}{a}$, giving your answer in the form $x + iy$. **[3]**

(Q4, June 2010)

25.

(i) Sketch on a single Argand diagram the loci given by

(a) $|z - 3 + 4i| = 5$, **[2]**

(b) $|z| = |z - 6|$. **[2]**

(ii) Indicate, by shading, the region of the Argand diagram for which

$$|z - 3 + 4i| \leqslant 5 \quad \text{and} \quad |z| \geqslant |z - 6|.$$ **[2]**

(Q6, June 2010)

26.

The complex number z, where $0 < \arg z < \frac{1}{2}\pi$, is such that $z^2 = 3 + 4\text{i}$.

(i) Use an algebraic method to find z. **[5]**

(ii) Show that $z^3 = 2 + 11\text{i}$. **[1]**

The complex number w is the root of the equation

$$w^6 - 4w^3 + 125 = 0$$

for which $-\frac{1}{2}\pi < \arg w < 0$.

(iii) Find w. **[5]**

(Q10, June 2010)

27.

The complex numbers z and w are given by $z = 4 + 3\text{i}$ and $w = 6 - \text{i}$. Giving your answers in the form $x + \text{i}y$ and showing clearly how you obtain them, find

(i) $3z - 4w$, **[2]**

(ii) $\dfrac{z^*}{w}$. **[4]**

(Q2, Jan 2011)

28.

(i) Sketch on a single Argand diagram the loci given by

(a) $|z| = |z - 8|$, **[2]**

(b) $\arg(z + 2\text{i}) = \frac{1}{4}\pi$. **[3]**

(ii) Indicate by shading the region of the Argand diagram for which

$$|z| \leqslant |z - 8| \quad \text{and} \quad 0 \leqslant \arg(z + 2\text{i}) \leqslant \tfrac{1}{4}\pi.$$ **[3]**

(Q6, Jan 2011)

29.

The complex number $1 + \mathrm{i}\sqrt{3}$ is denoted by a.

(i) Find $|a|$ and $\arg a$. **[2]**

(ii) Sketch on a single Argand diagram the loci given by $|z - a| = |a|$ and $\arg(z - a) = \frac{1}{2}\pi$. **[6]**

(Q5, June 2011)

30.

One root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real, is $16 - 30\mathrm{i}$.

(i) Write down the other root of the quadratic equation. **[1]**

(ii) Find the values of a and b. **[4]**

(iii) Use an algebraic method to solve the quartic equation $y^4 + ay^2 + b = 0$. **[7]**

(Q9, June 2011)

31.

The complex number $a + 5\mathrm{i}$, where a is positive, is denoted by z. Given that $|z| = 13$, find the value of a and hence find $\arg z$. **[4]**

(Q1, Jan 2012)

32.

Use an algebraic method to find the square roots of $3 + (6\sqrt{2})\mathrm{i}$. Give your answers in the form $x + \mathrm{i}y$, where x and y are exact real numbers. **[6]**

(Q3 Jan 2012)

33.

Sketch, on a single Argand diagram, the loci given by $|z - \sqrt{3} - \mathrm{i}| = 2$ and $\arg z = \frac{1}{6}\pi$. **[6]**

(Q6, Jan 2012)

34.

The complex numbers z and w are given by $z = 6 - \text{i}$ and $w = 5 + 4\text{i}$. Giving your answers in the form $x + \text{i}y$ and showing clearly how you obtain them, find

(i) $z + 3w$, **[2]**

(ii) $\frac{z}{w}$. **[3]**

(Q1, June 2012)

35.

The loci C_1 and C_2 are given by $|z - 3 - 4\text{i}| = 4$ and $|z| = |z - 8\text{i}|$ respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. **[6]**

(ii) Hence find the complex numbers represented by the points of intersection of C_1 and C_2. **[2]**

(iii) Indicate, by shading, the region of the Argand diagram for which

$$|z - 3 - 4\text{i}| \leqslant 4 \text{ and } |z| \geqslant |z - 8\text{i}|.$$ **[2]**

(Q7, June 2012)

36.

The complex number $2 - \text{i}$ is denoted by z.

(i) Find $|z|$ and $\arg z$. **[2]**

(ii) Given that $az + bz^* = 4 - 8\text{i}$, find the values of the real constants a and b. **[5]**

(Q3, Jan 2013)

37.

(i) Sketch on a single Argand diagram the loci given by

(a) $|z| = 2$, **[2]**

(b) $\arg(z - 3 - \text{i}) = \pi$. **[3]**

(ii) Indicate, by shading, the region of the Argand diagram for which

$$|z| \leqslant 2 \text{ and } 0 \leqslant \arg(z - 3 - \text{i}) \leqslant \pi.$$ **[2]**

(Q7, Jan 2013)

38.

The complex number $3 + a\text{i}$, where a is real, is denoted by z. Given that $\arg z = \frac{1}{6}\pi$, find the value of a and hence find $|z|$ and $z^* - 3$. **[6]**

(Q1, June 2013)

39.

Use an algebraic method to find the square roots of $11 + (12\sqrt{5})\text{i}$. Give your answers in the form $x + \text{i}y$, where x and y are exact real numbers. **[6]**

(Q3, June 2013)

40.

The complex number $7 + 3\text{i}$ is denoted by z. Find

(i) $|z|$ and $\arg z$, **[2]**

(ii) $\dfrac{z}{4-\text{i}}$, showing clearly how you obtain your answer. **[3]**

(Q4, Jan 2014)

41.

The loci C_1 and C_2 are given by $\arg(z-2-2\text{i}) = \frac{1}{4}\pi$ and $|z| = |z-10|$ respectively.

(i) Sketch on a single Argand diagram the loci C_1 and C_2. **[4]**

(ii) Indicate, by shading, the region of the Argand diagram for which

$$0 \leqslant \arg(z-2-2\text{i}) \leqslant \tfrac{1}{4}\pi \text{ and } |z| \geqslant |z-10|.$$ **[3]**

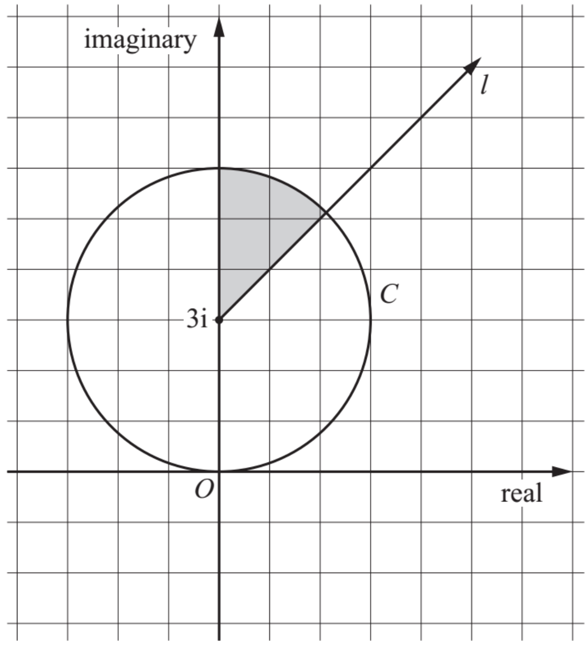
(Q7, June 2014)

42.

The complex number $x + \text{i}y$ is denoted by z. Express $3zz^* - |z|^2$ in terms of x and y. **[3]**

(Q1, June 2015)

43.



The Argand diagram above shows a half-line l and a circle C. The circle has centre 3i and passes through the origin.

(i) Write down, in complex number form, the equations of l and C. **[4]**

(ii) Write down inequalities that define the region shaded in the diagram. [The shaded region includes the boundaries.] **[3]**

(Q6, June 2013)

44.

The loci C_1 and C_2 are given by $|z+2| = 2$ and $\arg(z+2) = \frac{5}{6}\pi$ respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. **[4]**

(ii) Find the complex number represented by the intersection of C_1 and C_2. **[2]**

(iii) Indicate, by shading, the region of the Argand diagram for which

$$|z+2| \leqslant 2 \text{ and } \tfrac{5}{6}\pi \leqslant \arg(z+2) \leqslant \pi.$$ **[2]**

(Q5, June 2015)

45.

(i) Use an algebraic method to find the square roots of the complex number $5+12\mathrm{i}$. You must show sufficient working to justify your answers. **[5]**

(ii) Hence solve the quadratic equation $x^2-4x-1-12\mathrm{i}=0$. **[5]**

(Q7, June 2015)